HL Paper 1

Find the values of x for which the vectors $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix}$ are perpendicular, $0 \leqslant x \leqslant \frac{\pi}{2}$.

Markscheme

perpendicular when $\begin{pmatrix} 1 \\ 2\cos x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2\sin x \\ 1 \end{pmatrix} = 0$ ***(M1)***

$\Rightarrow -1 + 4\sin x\cos x = 0$ ***A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ ***M1***

$\Rightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$ ***A1A1***

Note: Accept answers in degrees.

[5 marks]

Examiners report

Most candidates realised that the scalar product should be used to solve this problem and many obtained the equation $4\sin x\cos x = 1$.

Candidates who failed to see that this could be written as $\sin 2x = 0.5$ usually made no further progress. The majority of those candidates who used this double angle formula carried on to obtain the solution $\frac{\pi}{12}$ but few candidates realised that $\frac{5\pi}{12}$ was also a solution.

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a. Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p. [3]

b. Hence find the value of $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$. [3]

Markscheme

a. attempt at use of $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ ***M1***

$\frac{1}{p} = \frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}} \left(=\frac{1}{3}\right)$ ***A1***

$p = 3$ ***A1***

Note: the value of p needs to be stated for the final mark.

[3 marks]

b. $\tan\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)\right) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$ ***M1A1***

$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \frac{\pi}{4}$ ***A1***

[3 marks]

Examiners report

a. Those candidates who used the addition formula for the tangent were usually successful.

b. Some candidates left their answer as the tangent of an angle, rather than the angle itself.

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a. Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to prove that $\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}},\ 0 \leqslant x \leqslant \pi$. [2]

b. Find a similar expression for $\sin \frac{1}{2}x,\ 0 \leqslant x \leqslant \pi$. [2]

c. Hence find the value of $\int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) \mathrm{d}x$. [4]

Markscheme

a. $\cos x = 2\cos^2\frac{1}{2}x - 1$

$\cos \frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \leqslant x \leqslant \pi$ ***R1***

$\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

b. $\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin \frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

c. $\sqrt{2}\int_0^{\frac{\pi}{2}} \cos \frac{1}{2}x + \sin \frac{1}{2}x\mathrm{d}x$ ***A1***

$= \sqrt{2}\left[2\sin \frac{1}{2}x - 2\cos \frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***A1***

$= 2\sqrt{2}$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

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a. Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. [2]

b. Hence find the value of $\cot \frac{\pi}{8}$ in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$. [3]

Markscheme

a. $\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}$ ***M1***

Note: Award ***M1*** for use of double angle formulae.

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$ ***A1***

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$ ***AG***

[2 marks]

b. $\tan\frac{\pi}{8} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ ***(M1)***

$\cot\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$ ***M1***

$= \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= 1+\sqrt{2}$ ***A1***

[3 marks]

Examiners report

a. The performance in this question was generally good with most candidates answering (a) well; (b) caused more difficulties, in particular the rationalization of the denominator. A number of misconceptions were identified, for example $\cot\frac{\pi}{8} = \tan\frac{8}{\pi}$.

b. The performance in this question was generally good with most candidates answering (a) well; (b) caused more difficulties, in particular the rationalization of the denominator. A number of misconceptions were identified, for example $\cot\frac{\pi}{8} = \tan\frac{8}{\pi}$.

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a. Show that $\cot\alpha = \tan\left(\frac{\pi}{2}-\alpha\right)$ for $0 < \alpha < \frac{\pi}{2}$. [1]

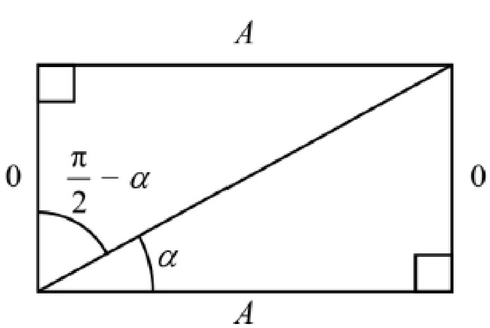
b. Hence find $\int_{\tan\alpha}^{\cot\alpha} \frac{1}{1+x^2}\mathrm{d}x,\ 0 < \alpha < \frac{\pi}{2}$. [4]

Markscheme

a. **EITHER**

use of a diagram and trig ratios

eg,



$\tan\alpha = \frac{O}{A} \Rightarrow \cot\alpha = \frac{A}{O}$

from diagram, $\tan\left(\frac{\pi}{2}-\alpha\right) = \frac{A}{O}$ ***R1***

OR

use of $\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)}=\frac{\cos\alpha}{\sin\alpha}$ ***R1***

THEN

$\cot\alpha=\tan\left(\frac{\pi}{2}-\alpha\right)$ ***AG***

[1 mark]

b. $\int_{\tan\alpha}^{\cot\alpha}\frac{1}{1+x^2}\mathrm{d}x=[\arctan x]_{\tan\alpha}^{\cot\alpha}$ ***(A1)***

Note: Limits (or absence of such) may be ignored at this stage.

$=\arctan(\cot\alpha)-\arctan(\tan\alpha)$ ***(M1)***

$=\frac{\pi}{2}-\alpha-\alpha$ ***(A1)***

$=\frac{\pi}{2}-2\alpha$ ***A1***

[4 marks]

Examiners report

a. This was generally well done.

b. This was generally well done. Some weaker candidates tried to solve part (b) through use of a substitution, though the standard result $\arctan x$ was well known. A small number used $\arctan x+c$ and went on to obtain an incorrect final answer.

In the triangle ABC, $\mathrm{AB}=2\sqrt{3}$, AC = 9 and $\mathrm{B\hat{A}C}=150^\circ$.

a. Determine BC, giving your answer in the form $k\sqrt{3}, k\in\mathbb{Z}^+$. [3]

b. The point D lies on (BC), and (AD) is perpendicular to (BC). Determine AD. [4]

Markscheme

a. $\mathrm{BC}^2=12+81+2\times2\sqrt{3}\times9\times\frac{\sqrt{3}}{2}=147$ ***M1A1***

$\mathrm{BC}=7\sqrt{3}$ ***A1***

[3 marks]

b. area of triangle $\mathrm{ABC}=\frac{1}{2}\times9\times2\sqrt{3}\times\frac{1}{2}\ \left(=\frac{9\sqrt{3}}{2}\right)$ ***M1A1***

therefore $\frac{1}{2}\times\mathrm{AD}\times7\sqrt{3}=\frac{9\sqrt{3}}{2}$ ***M1***

$\mathrm{AD}=\frac{9}{7}$ ***A1***

[4 marks]

Examiners report

[N/A]

b. [N/A]

Let $z = 1 - \cos 2\theta - \mathrm{i} \sin 2\theta,\ z \in \mathbb{C},\ 0 \leqslant \theta \leqslant \pi.$

a. Solve $2\sin(x + 60^\circ) = \cos(x + 30^\circ),\ 0^\circ \leqslant x \leqslant 180^\circ.$ [5]

b. Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}.$ [3]

c.i. Find the modulus and argument of z in terms of θ. Express each answer in its simplest form. [9]

c.ii. Hence find the cube roots of z in modulus-argument form. [5]

Markscheme

a. $2\sin(x + 60^\circ) = \cos(x + 30^\circ)$

$2(\sin x \cos 60^\circ + \cos x \sin 60^\circ) = \cos x \cos 30^\circ - \sin x \sin 30^\circ$ ***(M1)(A1)***

$2\sin x \times \frac{1}{2} + 2\cos x \times \frac{\sqrt{3}}{2} = \cos x \times \frac{\sqrt{3}}{2} - \sin x \times \frac{1}{2}$ ***A1***

$\Rightarrow \frac{3}{2}\sin x = -\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow \tan x = -\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x = 150^\circ$ ***A1***

[5 marks]

b. **EITHER**

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$ and

$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$ ***(A1)***

$\sin 105^\circ + \cos 105^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$ ***A1***

$= \frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$(\sin 105^\circ + \cos 105^\circ)^2 = \sin^2 105^\circ + 2\sin 105^\circ \cos 105^\circ + \cos^2 105^\circ$

$(\sin 105^\circ + \cos 105^\circ)^2 = 1 + \sin 210^\circ$ ***A1***

$= \frac{1}{2}$ ***A1***

$\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

c.i. **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1 - \cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1 - \cos 2\theta)}$ ***A1***

$= \sqrt{2(2\sin^2\theta)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\frac{\sin 2\theta}{1-\cos 2\theta}$ ***M1***

$= \frac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)$ ***A1***

$= \theta - \frac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)$ ***(A1)***

$= -2\mathrm{i}\sin\theta(\cos\theta + \mathrm{i}\sin\theta)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \frac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \frac{\pi}{2}$ ***A1***

[9 marks]

c.ii. attempt to apply De Moivre's theorem ***M1***

$(1 - \cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right)\right]$ ***A1A1A1***

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n = -1,\ 0,\ 1$. Equivalent forms are acceptable. ***A1***

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c.i. [N/A]
c.ii. [N/A]

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a. Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}$. [2]

b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. [2]

c. Use the principle of mathematical induction to prove that [9]

$\sin x+\sin 3x+\ldots+\sin(2n-1)x=\frac{1-\cos 2nx}{2\sin x},\ n\in\mathbb{Z}^+,\ x\neq k\pi$ where $k\in\mathbb{Z}$.

d. Hence or otherwise solve the equation $\sin x+\sin 3x=\cos x$ in the interval $0<x<\pi$. [6]

Markscheme

a. $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with \) + \) or $-$ signs.

[2 marks]

b. $\frac{1-\cos 2x}{2\sin x}\equiv\frac{1-(1-2\sin^2 x)}{2\sin x}$ ***M1***

$\equiv\frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv\sin x$ ***AG***

[2 marks]

c. let $\mathrm{P}(n):\sin x+\sin 3x+\ldots+\sin(2n-1)x\equiv\frac{1-\cos 2nx}{2\sin x}$

if $n=1$

$\mathrm{P}(1):\frac{1-\cos 2x}{2\sin x}\equiv\sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x+\sin 3x+\ldots+\sin(2k-1)x\equiv\frac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words “assume” and “true” appear. Do not award ***M1*** for “let $n=k$” only. Subsequent marks are independent of this ***M1***.

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1):\sin x+\sin 3x+\ldots+\sin(2k-1)x+\sin(2k+1)x\equiv\frac{1-\cos 2(k+1)x}{2\sin x}$

$LHS=\sin x+\sin 3x+\ldots+\sin(2k-1)x+\sin(2k+1)x$ ***M1***

$\equiv\frac{1-\cos 2kx}{2\sin x}+\sin(2k+1)x$ ***A1***

$\equiv\frac{1-\cos 2kx+2\sin x\sin(2k+1)x}{2\sin x}$

$\equiv\frac{1-\cos 2kx+2\sin x\cos x\sin 2kx+2\sin^2 x\cos 2kx}{2\sin x}$ ***M1***

$\equiv\frac{1-\left((1-2\sin^2 x)\cos 2kx-\sin 2x\sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv\frac{1-(\cos 2x\cos 2kx-\sin 2x\sin 2kx)}{2\sin x}$ ***A1***

$\equiv\frac{1-\cos(2kx+2x)}{2\sin x}$ ***A1***

$\equiv\frac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n=k$, then also true for $n=k+1$

as true for $n=1$ then true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

d. **EITHER**

$\sin x + \sin 3x = \cos x \Rightarrow \frac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1 - \cos 4x = 2\sin x\cos x,\ (\sin x \neq 0)$ ***A1***

$\Rightarrow 1 - (1 - 2\sin^2 2x) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \frac{1}{2}$ ***A1***

$2x = \pi,\ 2x = \frac{\pi}{6}$ and $2x = \frac{5\pi}{6}$

OR

$\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x\cos x = \cos x$ ***M1A1***

$\Rightarrow (2\sin 2x - 1)\cos x = 0,\ (\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ of $\cos x = 0$ ***A1***

$2x = \frac{\pi}{6},\ 2x = \frac{5\pi}{6}$ and $x = \frac{\pi}{2}$

THEN

$\therefore x = \frac{\pi}{2},\ x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ ***A1***

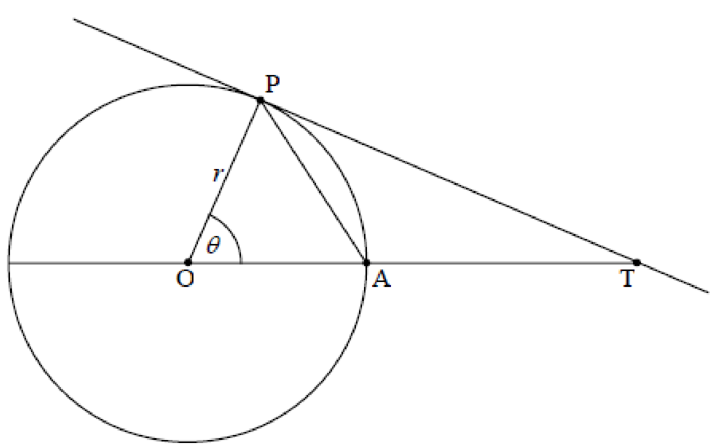
Note: Do not award the final ***A1*** if extra solutions are seen.

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The diagram shows a tangent, (TP) , to the circle with centre O and radius r . The size of $\mathrm{P\hat{O}A}$ is θ radians.



a. Find the area of triangle AOP in terms of r and θ . [1]

b. Find the area of triangle POT in terms of r and θ . [2]

c. Using your results from part (a) and part (b), show that $\sin \theta < \theta < \tan \theta$. [2]

Markscheme

a. area of AOP $= \frac{1}{2}r^2 \sin \theta$ ***A1***

[1 mark]

b. $\text{TP} = r \tan \theta$ ***(M1)***

area of POT $= \frac{1}{2}r(r \tan \theta)$

$= \frac{1}{2}r^2 \tan \theta$ ***A1***

[2 marks]

c. area of sector OAP $= \frac{1}{2}r^2\theta$ ***A1***

area of triangle OAP < area of sector OAP < area of triangle POT ***R1***

$\frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan \theta$

$\sin \theta < \theta < \tan \theta$ ***AG***

[2 marks]

Examiners report

a. The majority of candidates were able to find the area of Triangle *AOP* correctly. Most were then able to get an expression for the other triangle.

In the final section, few saw the connection between the area of the sector and the relationship.

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c. The majority of candidates were able to find the area of Triangle *AOP* correctly. Most were then able to get an expression for the other triangle.

In the final section, few saw the connection between the area of the sector and the relationship.

The first three terms of a geometric sequence are $\sin x,\ \sin 2x$ and $4 \sin x \cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio r.

(b) Find the set of values of x for which the geometric series $\sin x + \sin 2x + 4 \sin x \cos^2 x + \ldots$ converges.

Consider $x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.

Markscheme

(a) $\sin x,\ \sin 2x$ and $4 \sin x \cos^2 x$

$r = \frac{2 \sin x \cos x}{\sin x} = 2 \cos x$ **A1**

Note: Accept $\frac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \frac{1}{2}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \frac{\sin x}{1-2\cos x}$ ***M1***

$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1-2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$

$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{\sqrt{15}}{2}$ ***AG***

[3 marks]

Total [7 marks]

Examiners report

[N/A]

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi,\ \pi]$.

Markscheme

$(\sin 2x - \sin x) - (\cos 2x - \cos x) = 1$

attempt to use both double-angle formulae, in whatever form ***M1***

$(2\sin x\cos x - \sin x) - (2\cos^2 x - 1 - \cos x) = 1$

or $(2\sin x\cos x - \sin x) - (2\cos^2 x - \cos x) = 0$ for example ***A1***

Note: Allow any rearrangement of the above equations.

$\sin x(2\cos x - 1) - \cos x(2\cos x - 1) = 0$

$(\sin x - \cos x)(2\cos x - 1) = 0$ ***(M1)***

$\tan x = 1$ and $\cos x = \frac{1}{2}$ ***A1A1***

Note: These ***A*** marks are dependent on the ***M*** mark awarded for factorisation.

$x = -\frac{3\pi}{4},\ -\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{4}$ ***A2***

Note: Award ***A1*** for two correct answers, which could be for both tan or both cos solutions, for example.

[7 marks]

Examiners report

[N/A]

In triangle ABC, $\mathrm{BC} = \sqrt{3}\ \mathrm{cm}$, $\mathrm{A\hat{B}C} = \theta$ and $\mathrm{B\hat{C}A} = \frac{\pi}{3}$.

a. Show that length $\mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$. [4]

b. Given that AB has a minimum value, determine the value of θ for which this occurs. [4]

Markscheme

a. any attempt to use sine rule ***M1***

$\frac{\mathrm{AB}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$ ***A1***

$= \frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta}$ ***A1***

Note: Condone use of degrees.

$= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$ ***A1***

$\frac{\mathrm{AB}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}$

$\therefore \mathrm{AB} = \frac{3}{\sqrt{3}\cos\theta + \sin\theta}$ ***AG***

[4 marks]

b. **METHOD 1**

$(\mathrm{AB})' = \frac{-3\left(-\sqrt{3}\sin\theta + \cos\theta\right)}{\left(\sqrt{3}\cos\theta + \sin\theta\right)^2}$ ***M1A1***

setting $(\mathrm{AB})' = 0$ ***M1***

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 2

$\mathrm{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$

AB minimum when $\sin\left(\frac{2\pi}{3}-\theta\right)$ is maximum ***M1***

$\sin\left(\frac{2\pi}{3}-\theta\right)=1$ ***(A1)***

$\frac{2\pi}{3}-\theta=\frac{\pi}{2}$ ***M1***

$\theta=\frac{\pi}{6}$ ***A1***

METHOD 3

shortest distance from B to AC is perpendicular to AC ***R1***

$\theta=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$ ***M1A2***

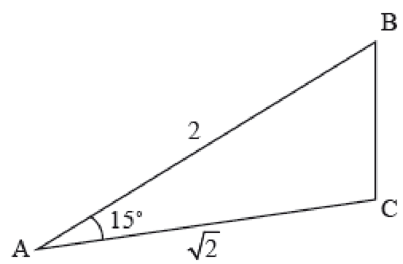
[4 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

The following diagram shows the triangle ABC where $\mathrm{AB}=2,\ \mathrm{AC}=\sqrt{2}$ and $\mathrm{B\hat{A}C}=15^\circ$.



a. Expand and simplify $\left(1-\sqrt{3}\right)^2$. [1]

b. By writing 15° as $60^\circ-45^\circ$ find the value of $\cos(15^\circ)$. [3]

c. Find BC in the form $a+\sqrt{b}$ where $a,\ b\in\mathbb{Z}$. [4]

Markscheme

a. $\left(1-\sqrt{3}\right)^2=4-2\sqrt{3}$ ***A1***

Note: Award ***A0*** for $1-2\sqrt{3}+3$.

[1 mark]

b. $\cos(60^\circ-45^\circ)=\cos(60^\circ)\cos(45^\circ)+\sin(60^\circ)\sin(45^\circ)$ ***M1***

$=\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\ \left(\text{or }\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right)$ ***(A1)***

$=\frac{\sqrt{2}+\sqrt{6}}{4}\ \left(\text{or }\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$ ***A1***

[3 marks]

c. $BC^2=2+4-2\times\sqrt{2}\times2\cos(15^\circ)$ ***M1***

$= 6 - \sqrt{2}\left(\sqrt{2} + \sqrt{6}\right)$

$= 4 - \sqrt{12}\ \left(= 4 - 2\sqrt{3}\right)$ ***A1***

$BC = \pm\left(1 - \sqrt{3}\right)$ ***(M1)***

$BC = -1 + \sqrt{3}$ ***A1***

Note: Accept $BC = \sqrt{3} - 1$.

Note: Award ***M1A0*** for $1 - \sqrt{3}$.

Note: Valid geometrical methods may be seen.

[4 marks]

Examiners report

a. The main error here was to fail to note the word 'simplify' in the question and some candidates wrote $1 + 3$ in their final answer rather than 4.

b. This was well done by the majority of candidates, though a few wrote $\cos(60 - 45) = \cos 60 - \cos 45$.

c. Candidates were able to use the cosine rule correctly but then failed to notice the result obtained was the same as that obtained in part (a).

The triangle ABC is equilateral of side 3 cm. The point D lies on [BC] such that BD = 1 cm.

Find $\cos \mathrm{D\hat{A}C}$.

Markscheme

METHOD 1

$\mathrm{AD}^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos 60°$ ***M1***

(or $\mathrm{AD}^2 = 1^2 + 3^2 - 2 \times 1 \times 3 \times \cos 60°$)

Note: ***M1*** for use of cosine rule with 60° angle.

$\mathrm{AD}^2 = 7$ ***A1***

$\cos \mathrm{D\hat{A}C} = \frac{9+7-4}{2\times 3\times\sqrt{7}}$ ***M1A1***

Note: **M1** for use of cosine rule involving $\mathrm{D\hat{A}C}$.

$= \frac{2}{\sqrt{7}}$ ***A1***

METHOD 2

let point E be the foot of the perpendicular from D to AC

EC = 1 (by similar triangles, or triangle properties) ***M1A1***

(or AE = 2)

$\mathrm{DE} = \sqrt{3}$ and $\mathrm{AD} = \sqrt{7}$ (by Pythagoras) ***(M1)A1***

$\cos \mathrm{D\hat{A}C} = \frac{2}{\sqrt{7}}$ ***A1***

Note: If first ***M1*** not awarded but remainder of the question is correct award ***M0A0M1A1A1***.

[5 marks]

Examiners report

[N/A]

Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

a. For what values of x does $f(x)$ not exist? [2]

b. Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$. [5]

Markscheme

a. $\cos x = 0,\ \sin x = 0$ ***(M1)***

$x = \frac{n\pi}{2}, n \in \mathbb{Z}$ ***A1***

b. **EITHER**

$\frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$ ***M1 A1***

$= \frac{\sin(3x-x)}{\frac{1}{2}\sin 2x}$ ***A1 A1***

$= 2$ ***A1***

OR

$\frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x} - \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x}$ ***M1***

$= \frac{2 \sin x \cos^2 x + 2\cos^2 x \sin x - \sin x}{\sin x} - \frac{2\cos^3 x - \cos x - \sin^2 x \cos x}{\cos x}$ ***A1 A1***

$= 4\cos^2 x - 1 - 2\cos^2 x + 1 + 2\sin^2 x$ ***A1***

$= 2\cos^2 x + 2\sin^2 x$

$= 2$ ***A1***

[5 marks]

Examiners report

a. Part (a) was well answered, although many candidates lost a mark through not giving sufficient solutions. It was rare for a student to receive no marks for part (b), but few solved the question by the easiest route, and as a consequence, there were frequently errors in the algebraic manipulation of the expression.

b. Part (a) was well answered, although many candidates lost a mark through not giving sufficient solutions. It was rare for a student to receive no marks for part (b), but few solved the question by the easiest route, and as a consequence, there were frequently errors in the algebraic manipulation of the expression.

A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.

Markscheme

METHOD 1

If the areas are in arithmetic sequence, then so are the angles. ***(M1)***

$\Rightarrow S_n = \frac{n}{2}(a+l) \Rightarrow \frac{12}{2}(\theta + 2\theta) = 18\theta$ ***M1A1***

$\Rightarrow 18\theta = 2\pi$ ***(A1)***

$\theta = \frac{\pi}{9}$ (accept 20°) ***A1***

[5 marks]

METHOD 2

$a_{12} = 2a_1$ ***(M1)***

$\frac{12}{2}(a_1 + 2a_1) = \pi r^2$ ***M1A1***

$3a_1 = \frac{\pi r^2}{6}$

$\frac{3}{2}r^2\theta = \frac{\pi r^2}{6}$ ***(A1)***

$\theta = \frac{2\pi}{18} = \frac{\pi}{9}$ (accept 20°) ***A1***

[5 marks]

METHOD 3

Let smallest angle = *a* , common difference = *d*

$a + 11d = 2a$ ***(M1)***

$a = 11d$ ***A1***

$S_n = \frac{12}{2}(2a + 11d) = 2\pi$ ***M1***

$6(2a + a) = 2\pi$ ***(A1)***

$18a = 2\pi$

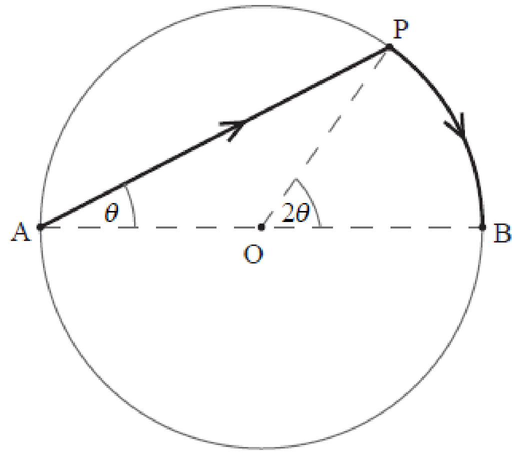
$a = \frac{\pi}{9}$ (accept 20°) ***A1***

[5 marks]

Examiners report

Stronger candidates had little problem with this question, but a significant minority of weaker candidates were unable to access the question or worked with area and very quickly became confused. Candidates who realised that the area of each sector was proportional to the angle usually gained the correct answer.

The diagram below shows a circular lake with centre O, diameter AB and radius 2 km.



Jorg needs to get from A to B as quickly as possible. He considers rowing to point P and then walking to point B. He can row at $3\ \mathrm{km\,h^{-1}}$ and walk at $6\ \mathrm{km\,h^{-1}}$. Let $\mathrm{P\hat{A}B} = \theta$ radians, and t be the time in hours taken by Jorg to travel from A to B.

a. Show that $t = \frac{2}{3}(2\cos\theta + \theta)$. [3]

b. Find the value of θ for which $\frac{\mathrm{d}t}{\mathrm{d}\theta} = 0$. [2]

c. What route should Jorg take to travel from A to B in the least amount of time? [3]

Give reasons for your answer.

Markscheme

a. angle APB is a right angle

$\Rightarrow \cos\theta = \frac{\mathrm{AP}}{4} \Rightarrow \mathrm{AP} = 4\cos\theta$ ***A1***

Note: Allow correct use of cosine rule.

arc $\mathrm{PB} = 2 \times 2\theta = 4\theta$ ***A1***

$t = \frac{\mathrm{AP}}{3} + \frac{\mathrm{PB}}{6}$ ***M1***

Note: Allow use of their AP and their PB for the ***M1***.

$\Rightarrow t = \frac{4\cos\theta}{3} + \frac{4\theta}{6} = \frac{4\cos\theta}{3} + \frac{2\theta}{3} = \frac{2}{3}(2\cos\theta + \theta)$ ***AG***

[3 marks]

b. $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{2}{3}(-2\sin\theta + 1)$ ***A1***

$\frac{2}{3}(-2\sin\theta + 1) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ (or 30 degrees) ***A1***

[2 marks]

c. $\frac{\mathrm{d}^2t}{\mathrm{d}\theta^2} = -\frac{4}{3}\cos\theta < 0 \quad \left(\text{at } \theta = \frac{\pi}{6}\right)$ ***M1***

$\Rightarrow t$ is maximized at $\theta = \frac{\pi}{6}$ ***R1***

time needed to walk along arc AB is $\frac{2\pi}{6}$ $(\approx 1\text{ hour})$

time needed to row from A to B is $\frac{4}{3}$ $(\approx 1.33\text{ hour})$

hence, time is minimized in walking from A to B ***R1***

[3 marks]

Examiners report

a. The fairly easy trigonometry challenged a large number of candidates.

b. Part (b) was very well done.

c. Satisfactory answers were very rarely seen for (c). Very few candidates realised that a minimum can occur at the beginning or end of an interval.

---

a. Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. [2]

b. Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3]

c. Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. [3]

d. **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. [6]

e. By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. [8]

Markscheme

a. $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\,(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$ ***(M1)***

$= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$ ***A1***

[2 marks]

b. from De Moivre's theorem

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ ***(M1)***

$\cos 3\theta + \mathrm{i}\sin 3\theta = \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$

equating real parts ***M1***

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right)$ ***A1***

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ ***AG***

Note: Do not award marks if part (a) is not used.

[3 marks]

c. $(\cos\theta + \mathrm{i}\sin\theta)^5 =$

$\cos^5\theta + 5\cos^4\theta\,(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2 + 10\cos^2\theta(\mathrm{i}\sin\theta)^3 + 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$ ***(A1)***

from De Moivre's theorem

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***M1***

$= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2$ ***A1***

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ***AG***

Note: If compound angles used in (b) and (c), then marks can be allocated in (c) only.

[3 marks]

d. $\cos 5\theta + \cos 3\theta + \cos\theta$

$= \left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right) + \left(4\cos^3\theta - 3\cos\theta\right) + \cos\theta = 0$ ***M1***

$16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0$ ***A1***

$\cos\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right) = 0$

$\cos\theta\left(4\cos^2\theta - 3\right)\left(4\cos^2\theta - 1\right) = 0$ ***A1***

$\therefore \cos\theta = 0; \pm\frac{\sqrt{3}}{2}; \pm\frac{1}{2}$ ***A1***

$\therefore \theta = \pm\frac{\pi}{6}; \pm\frac{\pi}{3}; \pm\frac{\pi}{2}$ ***A2***

[6 marks]

e. $\cos 5\theta = 0$

$5\theta = \ldots \frac{\pi}{2}; \left(\frac{3\pi}{2}; \frac{5\pi}{2}\right); \frac{7\pi}{2}; \ldots$ ***(M1)***

$\theta = \ldots \frac{\pi}{10}; \left(\frac{3\pi}{10}; \frac{5\pi}{10}\right); \frac{7\pi}{10}; \ldots$ ***(M1)***

Note: These marks can be awarded for verifications later in the question.

now consider $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ ***M1***

$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$

$\cos^2\theta = \frac{20\pm\sqrt{400-4(16)(5)}}{32}; \cos\theta = 0$ ***A1***

$\cos\theta = \pm\sqrt{\frac{20\pm\sqrt{400-4(16)(5)}}{32}}$

$\cos\frac{\pi}{10} = \sqrt{\frac{20+\sqrt{400-4(16)(5)}}{32}}$ since max value of cosine $\Rightarrow$ angle closest to zero ***R1***

$\cos\frac{\pi}{10} = \sqrt{\frac{4.5+4\sqrt{25-4(5)}}{4.8}} = \sqrt{\frac{5+\sqrt{5}}{8}}$ ***A1***

$\cos\frac{7\pi}{10} = -\sqrt{\frac{5-\sqrt{5}}{8}}$ ***A1A1***

[8 marks]

Examiners report

a. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

b. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

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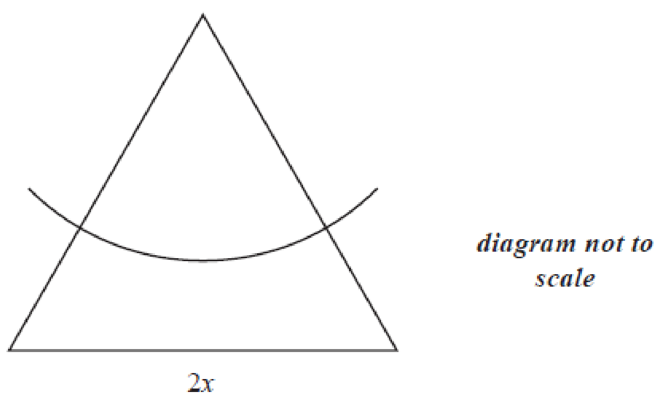
e. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

From a vertex of an equilateral triangle of side $2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.



Given that the areas of the two regions are equal, find the radius of the arc in terms of x.

Markscheme

area of triangle $= \frac{1}{2}(2x)^2 \sin\frac{\pi}{3}$ ***(M1)***

$= x^2\sqrt{3}$ ***A1***

Note: A 0.5 $\times$ base $\times$ height calculation is acceptable.

area of sector $= \frac{\theta}{2}r^2 = \frac{\pi}{6}r^2$ ***(M1)A1***

area of triangle is twice the area of the sector

$\Rightarrow 2\left(\frac{\pi}{6}r^2\right) = x^2\sqrt{3}$ ***M1***

$\Rightarrow r = x\sqrt{\frac{3\sqrt{3}}{\pi}}$ or equivalent ***A1***

[6 marks]

Examiners report

The majority of candidates obtained the correct answer. A small minority of candidates used degree measure rather than radian measure, or failed to notice that the triangle was equilateral.

The angle θ lies in the first quadrant and $\cos\theta = \frac{1}{3}$.

a. Write down the value of $\sin\theta$. [1]

b. Find the value of $\tan 2\theta$. [2]

c. Find the value of $\cos\left(\frac{\theta}{2}\right)$, giving your answer in the form $\frac{\sqrt{a}}{b}$ where a , $b \in \mathbb{Z}^+$. [3]

Markscheme

a. $\sin\theta = \frac{\sqrt{8}}{3}$ ***A1***

[1 mark]

b. $\tan 2\theta = \frac{2\times\sqrt{8}}{1-8} = -\frac{2\sqrt{8}}{7}$ $\left(-\frac{4\sqrt{2}}{7}\right)$ ***M1A1***

[2 marks]

c. $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\frac{1}{3}}{2} = \frac{2}{3}$ ***M1A1***

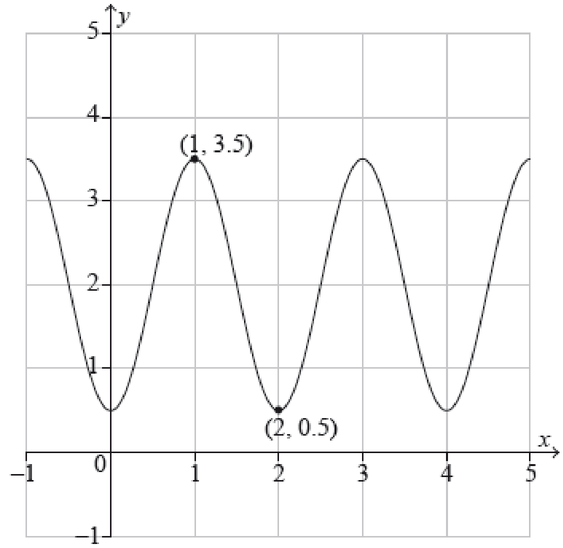
$\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{6}}{3}$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The following diagram shows the curve $y = a\sin(b(x+c)) + d$, where a, b, c and d are all positive constants. The curve has a maximum point at $(1,\ 3.5)$ and a minimum point at $(2,\ 0.5)$.



a. Write down the value of a and the value of d. [2]

b. Find the value of b. [2]

c. Find the smallest possible value of c, given $c > 0$. [2]

Markscheme

a. $a = 1.5$ $d = 2$ ***A1A1***

[2 marks]

b. $b = \frac{2\pi}{2} = \pi$ ***(M1)A1***

[2 marks]

c. attempt to solve an appropriate equation or apply a horizontal translation ***(M1)***

$c = 1.5$ ***A1***

Note: Do not award a follow through mark for the final ***A1***.

Award ***(M1)A0*** for $c = -0.5$.

[2 marks]

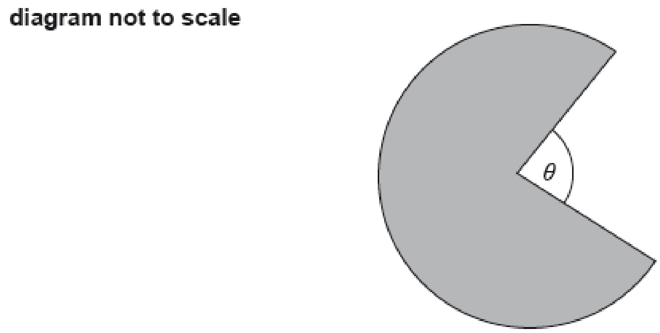
Examiners report

a. Parts (a) and (b) were largely done successfully, but there was still a large minority who did not score well, and this is something that teachers need to be aware of for the future.

b. Parts (a) and (b) were largely done successfully, but there was still a large minority who did not score well, and this is something that teachers need to be aware of for the future.

c. This part was less successfully done. Some attempted the question by putting in a point and solving the equation. Others did it through realizing it represented a horizontal translation. Of these many failed to heed the instruction (given in the stem of the question, as well as repeated in part (c)) that c had to be greater than zero.

The logo, for a company that makes chocolate, is a sector of a circle of radius 2 cm, shown as shaded in the diagram. The area of the logo is $3\pi\ \text{cm}^2$.



a. Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

b. Find the total length of the perimeter of the logo. [2]

Markscheme

a. **METHOD 1**

$\text{area} = \pi 2^2 - \frac{1}{2}2^2\theta \quad (= 3\pi)$ ***M1A1***

Note: Award ***M1*** for using area formula.

$\Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$ ***A1***

Note: Degrees loses final A1

METHOD 2

let $x = 2\pi - \theta$

$\text{area} = \frac{1}{2}2^2x \quad (= 3\pi)$ ***M1***

$\Rightarrow x = \frac{3}{2}\pi$ ***A1***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

METHOD 3

Area of circle is 4π ***A1***

Shaded area is $\frac{3}{4}$ of the circle ***(R1)***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

[3 marks]

b. $\text{arc length} = 2\frac{3\pi}{2}$ ***A1***

$\text{perimeter} = 2\frac{3\pi}{2} + 2 \times 2$

$= 3\pi + 4$ ***A1***

[2 marks]

Total [5 marks]

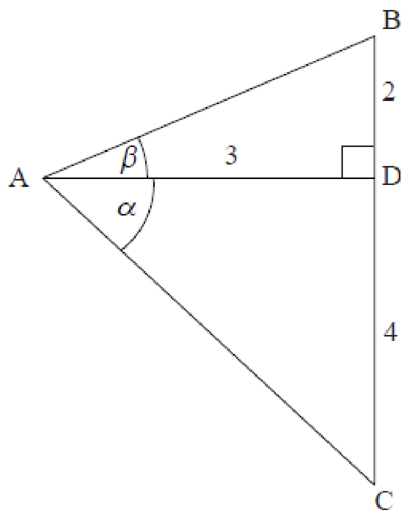
Examiners report

a. Good methods. Some candidates found the larger angle.

b. Generally good, some forgot the radii.

In the diagram below, AD is perpendicular to BC.

$\mathrm{CD} = 4$, $\mathrm{BD} = 2$ and $\mathrm{AD} = 3$. $\mathrm{C\hat{A}D} = \alpha$ and $\mathrm{B\hat{A}D} = \beta$.



Find the exact value of $\cos(\alpha - \beta)$.

Markscheme

METHOD 1

$\mathrm{AC} = 5$ and $\mathrm{AB} = \sqrt{13}$ (may be seen on diagram) ***(A1)***

$\cos\alpha = \frac{3}{5}$ **and** $\sin\alpha = \frac{4}{5}$ ***(A1)***

$\cos\beta = \frac{3}{\sqrt{13}}$ **and** $\sin\beta = \frac{2}{\sqrt{13}}$ ***(A1)***

Note: If only the two cosines are correctly given award ***(A1)(A1)(A0)***.

Use of $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ***(M1)***

$= \frac{3}{5} \times \frac{3}{\sqrt{13}} + \frac{4}{5} \times \frac{2}{\sqrt{13}}$ (substituting) ***M1***

$= \frac{17}{5\sqrt{13}}$ $\left(= \frac{17\sqrt{13}}{65}\right)$ ***A1 N1***

[6 marks]

METHOD 2

$\mathrm{AC} = 5$ amd $\mathrm{AB} = \sqrt{13}$ (may be seen on diagram) ***(A1)***

Use of $\cos(\alpha + \beta) = \frac{\mathrm{AC}^2 + \mathrm{AB}^2 - \mathrm{BC}^2}{2(\mathrm{AC})(\mathrm{AB})}$ ***(M1)***

$= \frac{25+13-36}{2\times5\times\sqrt{13}}$ $\left(= \frac{1}{5\sqrt{13}}\right)$ ***A1***

Use of $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$ ***(M1)***

$\cos\alpha=\frac{3}{5}$ **and** $\cos\beta=\frac{3}{\sqrt{13}}$ ***(A1)***

$\cos(\alpha-\beta)=\frac{17}{5\sqrt{13}}$ $\left(=2\times\frac{3}{5}\times\frac{3}{\sqrt{13}}-\frac{1}{5\sqrt{13}}\right)$ $\left(=\frac{17\sqrt{13}}{65}\right)$ ***A1 N1***

[6 marks]

Examiners report

Many candidates used a lot of space answering this question, but were generally successful. A few candidates incorrectly used the formula for the cosine of the difference of angles. An interesting alternative solution was noted, in which the side AB is reflected in AD and the required result follows from the use of the cosine rule.

Consider the curve defined by the equation $x^2+\sin y-xy=0$.

a. Find the gradient of the tangent to the curve at the point $(\pi,\ \pi)$. [6]

b. Hence, show that $\tan\theta=\frac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at $(\pi,\ \pi)$ and the line $y=x$. [3]

Markscheme

a. attempt to differentiate implicitly ***M1***

$2x+\cos y\frac{\mathrm{d}y}{\mathrm{d}x}-y-x\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***A1A1***

Note: ***A1*** for differentiating x^2 and $\sin y$; ***A1*** for differentiating xy.

substitute x and y by π ***M1***

$2\pi-\frac{\mathrm{d}y}{\mathrm{d}x}-\pi-\pi\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\pi}{1+\pi}$ ***M1A1***

Note: ***M1*** for attempt to make $\mathrm{d}y/\mathrm{d}x$ the subject. This could be seen earlier.

[6 marks]

b. $\theta=\frac{\pi}{4}-\arctan\frac{\pi}{1+\pi}$ (or seen the other way) ***M1***

$\tan\theta=\tan\left(\frac{\pi}{4}-\arctan\frac{\pi}{1+\pi}\right)=\frac{1-\frac{\pi}{1+\pi}}{1+\frac{\pi}{1+\pi}}$ ***M1A1***

$\tan\theta=\frac{1}{1+2\pi}$ ***AG***

[3 marks]

Examiners report

a. Part a) proved an easy 6 marks for most candidates, while the majority failed to make any headway with part b), with some attempting to find the equation of their line in the form $y=mx+c$. Only the best candidates were able to see their way through to the given answer.

b. Part a) proved an easy 6 marks for most candidates, while the majority failed to make any headway with part b), with some attempting to find the equation of their line in the form $y=mx+c$. Only the best candidates were able to see their way through to the given answer.

Show that $\frac{\cos A+\sin A}{\cos A-\sin A}=\sec 2A+\tan 2A$.

Markscheme

METHOD 1

$\frac{\cos A+\sin A}{\cos A-\sin A}=\sec 2A+\tan 2A$

consider right hand side

$\sec 2A+\tan 2A=\frac{1}{\cos 2A}+\frac{\sin 2A}{\cos 2A}$ ***M1A1***

$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

Note: Award ***A1*** for recognizing the need for single angles and ***A1*** for recognizing $\cos^2 A+\sin^2 A=1$.

$=\frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$=\frac{\cos A+\sin A}{\cos A-\sin A}$ ***AG***

METHOD 2

$\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$=\frac{1+\sin 2A}{\cos 2A}$ ***M1A1***

$=\sec 2A+\tan 2A$ ***AG***

[6 marks]

Examiners report

Solutions to this question were good in general with many candidates realising that multiplying the numerator and denominator by $(\cos A+\sin A)$ might be helpful.

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y$.

(b) Given $f(x)=\sin(x+\frac{\pi}{6})\sin(x-\frac{\pi}{6}),\ x\in[0,\ \pi]$, find the range of f.

(c) Given $g(x)=\csc(x+\frac{\pi}{6})\csc(x-\frac{\pi}{6}),\ x\in[0,\ \pi]\ ,\ x\neq\frac{\pi}{6},\ x\neq\frac{5\pi}{6}$, find the range of g.

Markscheme

(a) $\sin(x+y)\sin(x-y)$

$=(\sin x\cos y+\cos x\sin y)(\sin x\cos y-\cos x\sin y)$ ***M1A1***

$=\sin^2 x\cos^2 y+\sin x\sin y\cos x\cos y-\sin x\sin y\cos x\cos y-\cos^2 x\sin^2 y$

$=\sin^2 x\cos^2 y-\cos^2 x\sin^2 y$ ***A1***

$=\sin^2 x(1-\sin^2 y)-\sin^2 y(1-\sin^2 x)$ ***A1***

$=\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y$

$=\sin^2 x-\sin^2 y$ ***AG***

[4 marks]

(b) $f(x)=\sin^2 x-\frac{1}{4}$

range is $f\in\left[-\frac{1}{4},\ \frac{3}{4}\right]$ ***A1A1***

Note: Award ***A1*** for each end point. Condone incorrect brackets.

[2 marks]

(c) $g(x) = \frac{1}{\sin^2 x - \frac{1}{4}}$

range is $g \in \left]-\infty, -4\right] \cup \left[\frac{4}{3}, \infty\right[$ ***A1A1***

Note: Award ***A1*** for each part of range. Condone incorrect brackets.

[2 marks]

Total [8 marks]

Examiners report

Part a) often proved to be an easy 4 marks for candidates. A number were surprisingly content to gain the first 3 marks but were unable to make the final step by substituting $1 - \sin^2 y$ for $\cos^2 y$.

Parts b) and c) were more often than not, problematic. Some puzzling 'working' was often seen, with candidates making little headway. Otherwise good candidates were able to answer part b), though correct solutions for c) were a rarity. The range $g \in \left[-4, \frac{4}{3}\right]$ was sometimes seen, but gained no marks.

Consider the equation $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2},\ 0 < x < \frac{\pi}{2}$. Given that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

a. verify that $x = \frac{\pi}{12}$ is a solution to the equation; [3]

b. hence find the other solution to the equation for $0 < x < \frac{\pi}{2}$. [5]

Markscheme

a. **EITHER**

$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$ ***M1***

$= \frac{\sqrt{3}-1}{\frac{\sqrt{3}-1}{2\sqrt{2}}} + \frac{\sqrt{3}+1}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$ ***A1***

$= 2\sqrt{2} + 2\sqrt{2}$ ***A1***

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution ***AG***

OR

$\text{LHS} = \frac{\sqrt{3}-1}{\frac{\sqrt{6}-\sqrt{2}}{4}} + \frac{\sqrt{3}+1}{\frac{\sqrt{6}+\sqrt{2}}{4}}$ ***M1***

$= \frac{\left(\sqrt{3}-1\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) + \left(\sqrt{3}+1\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)}{\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)}$ ***A1***

$= 2\sqrt{18} - 2\sqrt{2}$ (or equivalent) ***A1***

$\text{LHS} = 4\sqrt{2} \Rightarrow x = \frac{\pi}{12}$ is a solution ***AG***

[3 marks]

b. $\frac{\sqrt{2}}{4}\left(\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x}\right) = 2 \Rightarrow \frac{\sin\frac{\pi}{12}}{\sin x} + \frac{\cos\frac{\pi}{12}}{\cos x} = 2$ ***M1***

$\frac{\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x}{\sin x\cos x} = 2$ ***M1***

$\sin\frac{\pi}{12}\cos x + \cos\frac{\pi}{12}\sin x = 2\sin x\cos x$

$\sin\left(\frac{\pi}{12} + x\right) = \sin 2x$ ***A1***

$\frac{\pi}{12} + x = \pi - 2x$ or $\pi - \left(\frac{\pi}{12} + x\right) = 2x$ ***(M1)***

$x = \frac{11\pi}{36}$ ***A1***

[5 marks]

Examiners report

a. This question proved to be the most problematic question in the paper.

Part (a) was generally well done, with competent fraction and surd manipulation seen successfully in leading to the given answer.

b. This question proved to be the most problematic question in the paper.

The number of scripts seen where part (b) was tackled with complete success numbered in the single figures; solutions were rarely if ever seen. Some candidates scored one mark by finding, or using, the common denominator $\sin x\cos x$.

Consider the functions $f(x) = \tan x,\ 0 \le\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1.$

a. Find an expression for $g \circ f(x)$, stating its domain. [2]

b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

c. Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

Markscheme

a. $g \circ f(x) = \frac{\tan x + 1}{\tan x - 1}$ ***A1***

$x \ne \frac{\pi}{4},\ 0 \le x < \frac{\pi}{2}$ ***A1***

[2 marks]

b. $\frac{\tan x + 1}{\tan x - 1} = \frac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ ***M1A1***

$= \frac{\sin x + \cos x}{\sin x - \cos x}$ ***AG***

[2 marks]

c. **METHOD 1**

$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ ***M1(A1)***

$\frac{dy}{dx} = \frac{(2\sin x\cos x - \cos^2 x - \sin^2 x) - (2\sin x\cos x + \cos^2 x + \sin^2 x)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \frac{-2}{1 - \sin 2x}$

Substitute $\frac{\pi}{6}$ into any formula for $\frac{dy}{dx}$ ***M1***

$\frac{-2}{1 - \sin\frac{\pi}{3}}$

$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$ ***A1***

$= \frac{-4}{2 - \sqrt{3}}$

$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ ***M1***

$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ ***A1***

METHOD 2

$\frac{dy}{dx} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$ ***M1A1***

$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$ ***A1***

$= \frac{-2\sec^2\frac{\pi}{6}}{\left(\tan\frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$ ***M1***

Note: Award ***M1*** for substitution $\frac{\pi}{6}$.

$\frac{-8}{\left(1 - \sqrt{3}\right)^2} = \frac{-8}{\left(4 - 2\sqrt{3}\right)}\frac{\left(4 + 2\sqrt{3}\right)}{\left(4 + 2\sqrt{3}\right)} = -8 - 4\sqrt{3}$ ***M1A1***

[6 marks]

d. Area $\left|\int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x}\,dx\right|$ ***M1***

$= \left|\left[\ln|\sin x - \cos x|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln|\sin 0 - \cos 0|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3} - 1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3} - 1}{2}\right) = \ln\left(\frac{2}{\sqrt{3} - 1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right)$ ***M1***

$= \ln\left(\sqrt{3} + 1\right)$ ***AG***

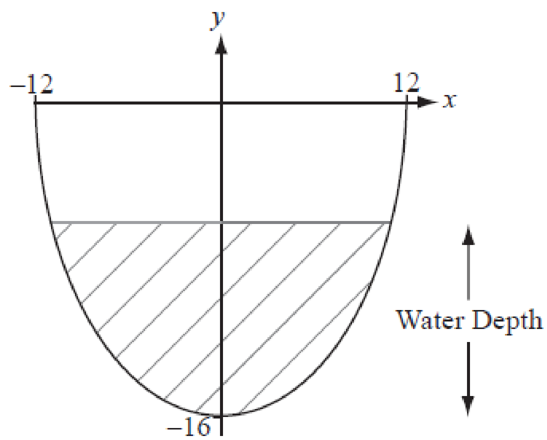
[6 marks]

Total [16 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The diagram below shows the boundary of the cross-section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.

Find the width of the water surface in the channel when the water depth is 10 cm.

Give your answer in the form $a\arccos b$ where $a,\ b \in \mathbb{R}$.

Markscheme

10 cm water depth corresponds to $16\sec\left(\frac{\pi x}{36}\right) - 32 = -6$ ***(A1)***

Rearranging to obtain an equation of the form $\sec\left(\frac{\pi x}{36}\right) = k$ or equivalent

i.e. making a trignometrical function the subject of the equation. ***M1***

$\cos\left(\frac{\pi x}{36}\right) = \frac{8}{13}$ ***(A1)***

$\frac{\pi x}{36} = \pm\arccos\frac{8}{13}$ ***M1***

$x = \pm\frac{36}{\pi}\arccos\frac{8}{13}$ ***A1***

Note: Do not penalise the omission of $\pm$.

Width of water surface is $\frac{72}{\pi}\arccos\frac{8}{13}$ (cm) ***R1*** ***N1***

Note: Candidate who starts with 10 instead of −6 has the potential to gain the two ***M1*** marks and the ***R1*** mark.

[6 marks]

Examiners report

This was a question in context which proved difficult for many candidates. Many appeared not to have fully comprehended the implications of the details of the diagram. A few candidates attempted integration, for no apparent reason.

In the triangle ABC, $\text{A}\hat{\text{B}}\text{C} = 90^\circ$, $\text{AC} = \sqrt{2}$ and AB = BC + 1.

a. Show that $\cos \hat{A} - \sin \hat{A} = \frac{1}{\sqrt{2}}$. [3]

b. By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle. [8]

c. Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin \hat{A} = \frac{\sqrt{6}-\sqrt{2}}{4}$. [6]

d. Hence, or otherwise, calculate the length of the perpendicular from B to [AC]. [4]

Markscheme

a. $\cos \hat{A} = \frac{\text{BA}}{\sqrt{2}}$ ***A1***

$\sin \hat{A} = \frac{\text{BC}}{\sqrt{2}}$ ***A1***

$\cos \hat{A} - \sin \hat{A} = \frac{\text{BA}-\text{BC}}{\sqrt{2}}$ ***R1***

$= \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

b. $\cos^2 \hat{A} - 2\cos \hat{A} \sin \hat{A} + \sin^2 \hat{A} = \frac{1}{2}$ ***M1A1***

$1 - 2\sin \hat{A} \cos \hat{A} = \frac{1}{2}$ ***M1A1***

$\sin 2\hat{A} = \frac{1}{2}$ ***M1***

$2\hat{A} = 30^\circ$ ***A1***

angles in the triangle are 15° and 75° ***A1A1***

Note: Accept answers in radians.

[8 marks]

c. $\text{BC}^2 + (\text{BC} + 1)^2 = 2$ ***M1A1***

$2\text{BC}^2 + 2\text{BC} - 1 = 0$ ***A1***

$\text{BC} = \frac{-2+\sqrt{12}}{4} \left(= \frac{\sqrt{3}-1}{2}\right)$ ***M1A1***

$\sin \hat{A} = \frac{\text{BC}}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ ***A1***

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ***AG***

[6 marks]

d. **EITHER**

$h = \text{AB}\sin\hat{A}$ ***M1***

$= (\text{BC} + 1) \sin \hat{A}$ ***A1***

$= \frac{\sqrt{3}+1}{2} \times \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}}{4}$ ***M1A1***

OR

$\frac{1}{2}AB.BC = \frac{1}{2}AC.h$ ***M1***

$\frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt{3+1}}{2} = \sqrt{2h}$ ***A1***

$\frac{2}{4} = \sqrt{2}h$ ***M1***

$h = \frac{1}{2\sqrt{2}}$ ***A1***

[4 marks]

Examiners report

a. Many good solutions to this question, although some students incorrectly stated the value of $\arcsin\left(\frac{1}{2}\right)$. A surprising number of students had greater difficulties with part (d).

b. Many good solutions to this question, although some students incorrectly stated the value of $\arcsin\left(\frac{1}{2}\right)$. A surprising number of students had greater difficulties with part (d).

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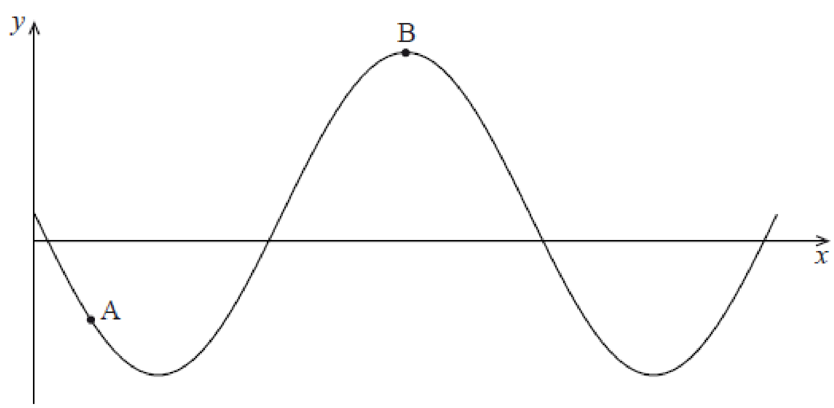
d. Many good solutions to this question, although some students incorrectly stated the value of $\arcsin\left(\frac{1}{2}\right)$. A surprising number of students had greater difficulties with part (d).

The diagram below shows a curve with equation $y = 1 + k\sin x$, defined for $0 \leqslant x \leqslant 3\pi$.



The point A $\left(\frac{\pi}{6}, -2\right)$ lies on the curve and $\mathrm{B}(a,\ b)$ is the maximum point.

(a) Show that $k = -6$.

(b) Hence, find the values of a and b .

Markscheme

(a) $-2 = 1 + k\sin\left(\frac{\pi}{6}\right)$ ***M1***

$-3 = \frac{1}{2}k$ ***A1***

$k = -6$ ***AG*** ***N0***

(b) **METHOD 1**

maximum $\Rightarrow \sin x = -1$ ***M1***

$a = \frac{3\pi}{2}$ ***A1***

$b = 1 - 6(-1)$

$= 7$ ***A1*** ***N2***

METHOD 2

$y' = 0$ ***M1***

$k\cos x = 0 \Rightarrow x = \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots$

$a = \frac{3\pi}{2}$ ***A1***

$b = 1 - 6(-1)$

$= 7$ ***A1*** ***N2***

Note: Award ***A1A1*** for $\left(\frac{3\pi}{2},\ 7\right)$.

[5 marks]

Examiners report

This was the most successfully answered question in the paper. Part (a) was done well by most candidates. In part (b), a small number of candidates used knowledge about transformations of functions to identify the coordinates of B. Most candidates used differentiation.

---

a. (i) Show that $(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta},\quad \cos\theta \ne 0.$ [10]

(ii) Hence verify that $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation $(1+z)^4 + (1-z)^4 = 0,\quad z \in \mathbb{C}.$

(iii) State another root of the equation $(1+z)^4 + (1-z)^4 = 0,\quad z \in \mathbb{C}.$

b. (i) Use the double angle identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1.$ [13]

(ii) Show that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$

(iii) Hence find the value of $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x.$

Markscheme

a. (i) **METHOD 1**

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \left(1 + \mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n + \left(1 - \mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n$ ***M1***

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem ***(M1)***

$\left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta + i\sin n\theta}{\cos^n\theta}$ ***A1*** recognition that $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$ ***(R1)***

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$\left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta - i\sin n\theta}{\cos^n\theta}$ ***A1***

$(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

METHOD 2

$(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = (1+\mathrm{i}\tan\theta)^n + (1+\mathrm{i}\tan(-\theta))^n$ ***(M1)***

$= \frac{(\cos\theta + i\sin\theta)^n}{\cos^n\theta} + \frac{(\cos(-\theta) + i\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$= \frac{1}{\cos^n\theta}(\cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta))$ ***A1***

$= \frac{2\cos n\theta}{\cos^2\theta}$ as $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$ ***R1AG***

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4 + \left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4 = \frac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$ ***(A1)***

$= \frac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$ ***A1***

$= 0$ as $\cos\frac{3\pi}{2} = 0$ ***R1***

Note: The above working could involve theta and the solution of $\cos(4\theta) = 0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation ***AG***

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$ ***A1***

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.

Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

[10 marks]

b. (i) $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$ ***A1***

let $t = \tan\frac{\pi}{8}$

attempting to solve $t^2 + 2t - 1 = 0$ for t ***M1***

$t = -1 \pm \sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8} > 0$ ***R1***

$\tan\frac{\pi}{8} = \sqrt{2} - 1$ ***AG***

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\mathrm{d}x$ ***A1***

$= [4\sin 2x - 8x + 2\tan x]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]

Examiners report

a. Fairly successful.

b. (i) Most candidates attempted to use the hint. Those who doubled the angle were usually successful – but many lost the final mark by not giving a convincing reason to reject the negative solution to the intermediate quadratic equation. Those who halved the angle got nowhere.

(ii) The majority of candidates obtained full marks.

(iii) This was poorly answered, few candidates realising that part of the integrand could be re-expressed using $\frac{1}{\cos^2 x} = \sec^2 x$, which can be immediately integrated.

In the triangle PQR, PQ = 6 , PR = k and $\mathrm{P\hat{Q}R} = 30^\circ$.

a. For the case $k = 4$, find the two possible values of QR. [4]

b. Determine the values of k for which the conditions above define a unique triangle. [3]

Markscheme

a. attempt to apply cosine rule ***M1***

$4^2 = 6^2 + \mathrm{QR}^2 - 2 \cdot \mathrm{QR} \cdot 6\cos 30^\circ$ (**or** $\mathrm{QR}^2 - 6\sqrt{3}\,\mathrm{QR} + 20 = 0$) ***A1***

$\mathrm{QR} = 3\sqrt{3} + \sqrt{7}$ or $\mathrm{QR} = 3\sqrt{3} - \sqrt{7}$ ***A1A1***

[4 marks]

b. **METHOD 1**

$k \geqslant 6$ ***A1***

$k = 6\sin 30^\circ = 3$ ***M1A1***

Note: The ***M1*** in (b) is for recognizing the right-angled triangle case.

METHOD 2

$k \geqslant 6$ ***A1***

use of discriminant: $108-4(36-k^2)=0$ ***M1***
$k = 3$ ***A1***
Note: $k = \pm 3$ is ***M1A0.***

[3 marks]

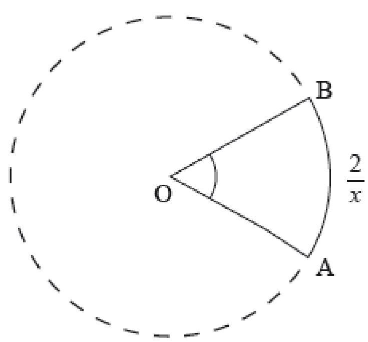
Examiners report

a. Candidates using the sine rule here made little or no progress. With the cosine rule, the two values are obtained quite quickly, which was the case for a majority of candidates. A small number were able to write down the correct quadratic equation to be solved, but then made arithmetical errors en route to their final solution(s). Part b) was often left blank. The better candidates were able to deduce $k = 3$, though the solution $k \geqslant 6$ was rarely, if at all, seen by examiners.

b. Candidates using the sine rule here made little or no progress. With the cosine rule, the two values are obtained quite quickly, which was the case for a majority of candidates. A small number were able to write down the correct quadratic equation to be solved, but then made arithmetical errors en route to their final solution(s). Part b) was often left blank. The better candidates were able to deduce $k = 3$, though the solution $k \geqslant 6$ was rarely, if at all, seen by examiners.

The following diagram shows a sector of a circle where $\mathrm{A\hat{O}B}=x$ radians and the length of the arc $\mathrm{AB}=\frac{2}{x}$ cm.

Given that the area of the sector is $16\ \mathrm{cm}^2$, find the length of the arc AB.



Markscheme

arc length $=\frac{2}{x}=rx\quad\left(\Rightarrow r=\frac{2}{x^2}\right)$ ***M1***

$16=\frac{1}{2}\left(\frac{2}{x^2}\right)^2 x\quad\left(\Rightarrow\frac{2}{x^3}=16\right)$ ***M1***

Note: Award ***M1***s for attempts at the use of arc-length and sector-area formulae.

$x=\frac{1}{2}$ ***A1***

arc length $=4$ (cm) ***A1***

[4 marks]

Examiners report

[N/A]

Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of $\sin 2\alpha$.

Markscheme

$\sin\alpha = \sqrt{1 - \left(-\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$ ***(M1)A1***
attempt to use double angle formula ***M1***
$\sin 2\alpha = 2\frac{\sqrt{7}}{4}\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$ ***A1***
Note: $\frac{\sqrt{7}}{4}$ seen would normally be awarded ***M1A1***.

[4 marks]

Examiners report

Many candidates scored full marks on this question, though their explanations for part a) often lacked clarity. Most preferred to use some kind of right-angled triangle rather than (perhaps in this case) the more sensible identity $\sin^2\alpha + \cos^2\alpha = 1$.

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ satisfy the equation $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$. Show that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Markscheme

taking cross products with $\boldsymbol{a}$, ***M1***

$\boldsymbol{a} \times (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{0} = \boldsymbol{0}$ ***A1***

using the algebraic properties of vectors and the fact that $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$, ***M1***

$\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{0}$ ***A1***

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}$ ***AG***

taking cross products with $\boldsymbol{b}$, ***M1***

$\boldsymbol{b} \times (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{0}$

$\boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{0}$ ***A1***

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$ ***AG***

this completes the proof

[6 marks]

Examiners report

[N/A]

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$
$g(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

a. Sketch the graph of $y = h(x)$. [2]

b. Find an expression for the composite function $h \circ g(x)$ and state its domain. [2]

c. Given that $f(x) = h(x) + h \circ g(x)$, [7]

(i) find $f'(x)$ in simplified form;

(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

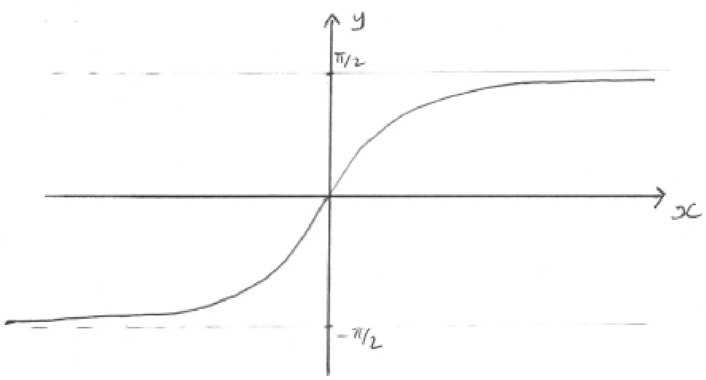
d. Nigel states that f is an odd function and Tom argues that f is an even function. [3]

(i) State who is correct and justify your answer.

(ii) Hence find the value of $f(x)$ for $x < 0$.

Markscheme

a.



A1A1

Note: ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

b. $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of $g : x \in \circ,\ x \neq 0$ ***A1***

[2 marks]

c. (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ ***M1A1***

$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$ ***(A1)***

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$

$= 0$ ***A1***

(ii) **METHOD 1**

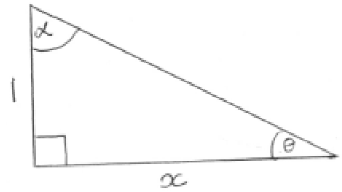
f is a constant ***R1***

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ ***M1A1***

$= \frac{\pi}{2}$ ***AG***

METHOD 2



from diagram

$\theta = \arctan \frac{1}{x}$ ***A1***

$\alpha = \arctan x$ ***A1***

$\theta + \alpha = \frac{\pi}{2}$ ***R1***

hence $f(x) = \frac{\pi}{2}$ ***AG***

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ ***M1***

$= \frac{x+\frac{1}{x}}{1-x\left(\frac{1}{x}\right)}$ ***A1***

denominator = 0, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) ***R1***

[7 marks]

d. (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore *f* is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

---

a. Show that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. [1]

b. Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

Markscheme

a. $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

b. consider $n = 1,\ f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x} \quad \left(= a\left(a^k \cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1} \sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1} \sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior ***M*** and ***R*** marks have been awarded.

[7 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$.

Markscheme

METHOD 1

use of $\sec^2 x = \tan^2 x + 1$ ***M1***

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$ ***(M1)***

$\tan x = -1$ ***A1***

$x = \frac{3\pi}{4},\ \frac{7\pi}{4}$ ***A1A1***

METHOD 2

$\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x} = 0$ ***M1***

$1 + 2\sin x\cos x = 0$

$\sin 2x = -1$ ***M1A1***

$2x = \frac{3\pi}{2},\ \frac{7\pi}{2}$

$x = \frac{3\pi}{4},\ \frac{7\pi}{4}$ ***A1A1***

Note: Award ***A1A0*** if extra solutions given or if solutions given in degrees (or both).

[5 marks]

Examiners report

[N/A]

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+,\ \sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2},\ 0 < x < \pi$.

Markscheme

(a) $\sin(2n+1)x\cos x - \cos(2n+1)x\sin x = \sin(2n+1)x - x$ ***M1A1***

$= \sin 2nx$ ***AG***

[2 marks]

(b) if $n = 1$ ***M1***

LHS $= \cos x$

RHS $= \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$ ***M1***

so LHS = RHS and the statement is true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

Note: Only award ***M1*** if the word **true** appears.

Do **not** award ***M1*** for 'let $n = k$' only.

Subsequent marks are independent of this ***M1***.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \frac{\sin 2kx}{2\sin x}$

if $n = k + 1$ then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ ***M1***

$= \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x$ ***A1***

$= \frac{\sin 2kx + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x}$ ***A1***

$= \frac{\sin(2k+2)x}{2\sin x}$ ***M1***

$= \frac{\sin 2(k+1)x}{2\sin x}$ ***A1***

so if true for $n = k$, then also true for $n = k + 1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Final ***R1*** is independent of previous work.

[12 marks]

(c) $\frac{\sin 4x}{2\sin x} = \frac{1}{2}$ ***M1A1***

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \frac{\pi}{5}$ ***A1***

$4x = 2\pi + x \Rightarrow x = \frac{2\pi}{3}$ ***A1***

$4x = 3\pi - x \Rightarrow x = \frac{3\pi}{5}$ ***A1***

for not including any answers outside the domain ***R1***

Note: Award the first ***M1A1*** for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\left(-\frac{1}{2}, \frac{1 \pm \sqrt{5}}{4}\right)$.

[6 marks]

Total [20 marks]

Examiners report

This question showed the weaknesses of many candidates in dealing with formal proofs and showing their reasoning in a logical manner. In part (a) just a few candidates clearly showed the result and part (b) showed that most candidates struggle with the formality of a proof by induction. The logic of many solutions was poor, though sometimes contained correct trigonometric work. Very few candidates were successful in answering part (c) using the unit circle. Most candidates attempted to manipulate the equation to obtain a cubic equation but made little progress. A few candidates guessed $\frac{2\pi}{3}$ as a solution but were not able to determine the other solutions.

A triangle has sides of length $(n^2 + n + 1)$, $(2n + 1)$ and $(n^2 - 1)$ where $n > 1$.

(a) Explain why the side $(n^2 + n + 1)$ must be the longest side of the triangle.

(b) Show that the largest angle, θ, of the triangle is 120°.

Markscheme

(a) a reasonable attempt to show either that $n^2 + n + 1 > 2n + 1$ or

$n^2 + n + 1 > n^2 - 1$ ***M1***

complete solution to each inequality ***A1A1***

(b) $\cos\theta = \frac{(2n+1)^2 + (n^2-1)^2 - (n^2+n+1)^2}{2(2n+1)(n^2-1)}$ ***M1A1***

$= \frac{-2n^3 - n^2 + 2n + 1}{2(2n+1)(n^2-1)}$ ***M1***

$= -\frac{(n-1)(n+1)(2n+1)}{2(2n+1)(n^2-1)}$ ***A1***

$= -\frac{1}{2}$ ***A1***

$\theta = 120^\circ$ ***AG***

[8 marks]

Examiners report

There were very few complete and accurate answers to part a). The most common incorrect response was to state the triangle inequality and feel that this was sufficient.

Many substituted a particular value for n and illustrated the result. Most students recognised the need for the Cosine rule and applied it correctly. Many then expanded and simplified to the correct answer. There was significant fudging in the middle on some papers. There were many good responses to this question.

Given that $\sin x + \cos x = \frac{2}{3}$, find $\cos 4x$.

Markscheme

$\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{4}{9}$ ***(M1)(A1)***

using $\sin^2 x + \cos^2 x = 1$ ***(M1)***

$2\sin x\cos x = -\frac{5}{9}$

using $2\sin x\cos x = \sin 2x$ ***(M1)***

$\sin 2x = -\frac{5}{9}$

$\cos 4x = 1 - 2\sin^2 2x$ ***M1***

Note: Award this ***M1*** for decomposition of cos 4*x* using double angle formula anywhere in the solution.

$= 1 - 2 \times \frac{25}{81}$

$= \frac{31}{81}$ ***A1***

[6 marks]

Examiners report

[N/A]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0^\circ \le x < 360^\circ$.

Markscheme

$\tan x + \tan 2x = 0$

$\tan x + \frac{2\tan x}{1-\tan^2 x} = 0$ ***M1***

$\tan x - \tan^3 x + 2\tan x = 0$ ***A1***

$\tan x(3 - \tan^2 x) = 0$ ***(M1)***

$\tan x = 0 \Rightarrow x = 0,\ x = 180^\circ$ ***A1***

Note: If $x = 360^\circ$ seen anywhere award ***A0***

$\tan x = \sqrt{3} \Rightarrow x = 60^\circ,\ 240^\circ$ ***A1***

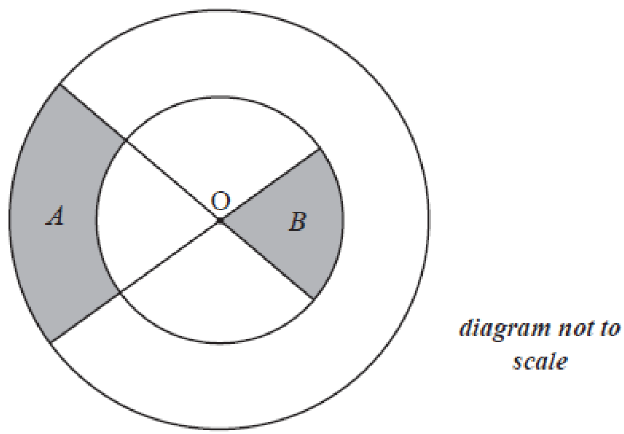
$\tan x = -\sqrt{3} \Rightarrow x = 120^\circ,\ 300^\circ$ ***A1***

[6 marks]

Examiners report

[N/A]

The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius R and the inner circle has radius r .



Consider the shaded regions with areas A and B . Given that $A : B = 2 : 1$, find the **exact** value of the ratio $R : r$.

Markscheme

$A = \frac{\theta}{2}(R^2 - r^2)$ ***A1***

$B = \frac{\theta}{2}r^2$ ***A1***

from $A : B = 2 : 1$, we have $R^2 - r^2 = 2r^2$ ***M1***

$R = \sqrt{3}r$ ***(A1)***

hence exact value of the ratio $R : r$ is $\sqrt{3} : 1$ ***A1*** ***N0***

[5 marks]

Examiners report

This question was successfully answered by most candidates using a variety of correct approaches. A few candidates, however, did not use a parameter for the angle, but instead substituted an angle directly, e.g., $\frac{\pi}{2}$ or $\frac{\pi}{4}$.

Consider the following system of equations:

$$x + y + z = 1$$

$$2x + 3y + z = 3$$

$$x + 3y - z = \lambda$$

where $\lambda \in \mathbb{R}$.

a. Show that this system does not have a unique solution for any value of λ . [4]

b. (i) Determine the value of λ for which the system is consistent. [4]

(ii) For this value of λ , find the general solution of the system.

Markscheme

a. using row operations, ***M1***

to obtain 2 equations in the same 2 variables ***A1A1***

for example $y - z = 1$

$2y - 2z = \lambda - 1$

the fact that one of the left hand sides is a multiple of the other left hand side indicates that the equations do not have a unique solution, or equivalent ***R1AG***

[4 marks]

b. (i) $\lambda = 3$ ***A1***

(ii) put $z = \mu$ ***M1***

then $y = 1 + \mu$ ***A1***

and $x = -2\mu$ or equivalent ***A1***

[4 marks]

Examiners report

a. [N/A]

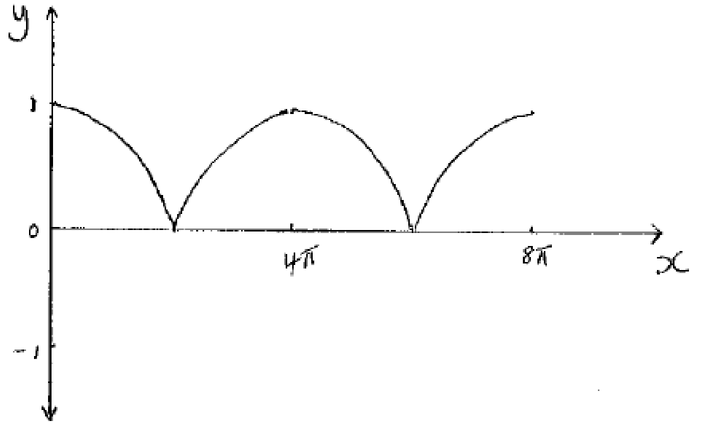
b. [N/A]

---

a. Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$. [2]

b. Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$. [3]

Markscheme

a. 

A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

b. $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$
$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$
$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

In triangle ABC, AB = 9 cm , AC = 12 cm , and $\hat{B}$ is twice the size of $\hat{C}$.

Find the cosine of $\hat{C}$.

Markscheme

$\frac{9}{\sin C} = \frac{12}{\sin B}$ ***(M1)***

$\frac{9}{\sin C} = \frac{12}{\sin 2C}$ ***A1***

Using double angle formula $\frac{9}{\sin C} = \frac{12}{2\sin C\cos C}$ ***M1***

$\Rightarrow 9(2\sin C\cos C) = 12\sin C$

$\Rightarrow 6\sin C(3\cos C-2)=0$ or equivalent ***(A1)***

$(\sin C\neq 0)$

$\Rightarrow \cos C=\frac{2}{3}$ ***A1***

[5 marks]

Examiners report

There were many totally correct solutions to this question, but again a significant minority did not make much progress. The most common reasons for this were that candidates immediately assumed that because the question asked for the cosine of $\hat{C}$ that they should use the cosine rule, or they did not draw a diagram and then confused which angles were opposite which sides.

Consider $w=2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

These four points form the vertices of a quadrilateral, *Q*.

a.i. Express w^2 and w^3 in modulus-argument form. [3]

a.ii. Sketch on an Argand diagram the points represented by w^0 , w^1 , w^2 and w^3. [2]

b. Show that the area of the quadrilateral *Q* is $\frac{21\sqrt{3}}{2}$. [3]

c. Let $z=2\left(\cos\frac{\pi}{n}+\mathrm{i}\sin\frac{\pi}{n}\right),\ n\in\mathbb{Z}^{+}$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2,\ \dots,\ z^n$ form the vertices of a polygon P_n. [6]

Show that the area of the polygon P_n can be expressed in the form $a\left(b^n-1\right)\sin\frac{\pi}{n}$, where $a,\ b\in\mathbb{R}$.

Markscheme

a.i. $w^2=4\mathrm{cis}\left(\frac{2\pi}{3}\right)$; $w^3=8\mathrm{cis}\left(\pi\right)$ ***(M1)A1A1***

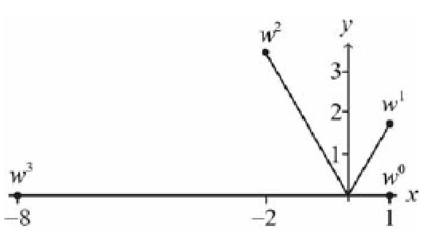
Note: Accept Euler form.

Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

[3 marks]

a.ii.



A1A1

[2 marks]

b. use of area = $\frac{1}{2}ab\ \sin C$ ***M1***

$\frac{1}{2}\times 1\times 2\times\sin\frac{\pi}{3}+\frac{1}{2}\times 2\times 4\times\sin\frac{\pi}{3}+\frac{1}{2}\times 4\times 8\times\sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

c. $\frac{1}{2} \times 2^0 \times 2^1 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^1 \times 2^2 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^2 \times 2^3 \times \sin\frac{\pi}{n} + \ldots + \frac{1}{2} \times 2^{n-1} \times 2^n \times \sin\frac{\pi}{n}$ ***M1A1***

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n} \times \left(2^0 + 2^2 + 2^4 + \ldots + 2^{n-2}\right)$

identifying a geometric series with common ratio $2^2 (= 4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4} \times \sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}(4^n - 1)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]

Let $a = \sin b,\ 0 < b < \frac{\pi}{2}$.

Find, in terms of *b*, the solutions of $\sin 2x = -a,\ 0 \leqslant x \leqslant \pi$.

Markscheme

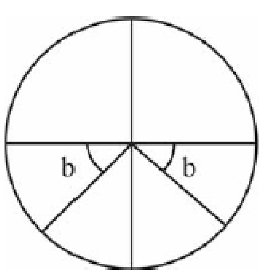
$\sin 2x = -\sin b$

EITHER

$\sin 2x = \sin(-b)$ or $\sin 2x = \sin(\pi + b)$ or $\sin 2x = \sin(2\pi - b) \ldots$ ***(M1)(A1)***

Note: Award ***M1*** for any one of the above, ***A1*** for having final two.

OR



(M1)(A1)

Note: Award ***M1*** for one of the angles shown with b clearly labelled, ***A1*** for both angles shown. Do not award ***A1*** if an angle is shown in the second quadrant and subsequent ***A1*** marks not awarded.

THEN

$2x = \pi + b$ or $2x = 2\pi - b$ ***(A1)(A1)***

$x = \frac{\pi}{2} + \frac{b}{2},\ x = \pi - \frac{b}{2}$ ***A1***

[5 marks]

Examiners report

[N/A]

---

a. Find all values of *x* for $0.1 \leqslant x \leqslant 1$ such that $\sin(\pi x^{-1}) = 0$. [2]

b. Find $\int_{\frac{1}{n+1}}^{\frac{1}{n}} \pi x^{-2} \sin(\pi x^{-1}) \mathrm{d}x$, showing that it takes different integer values when *n* is even and when *n* is odd. [3]

c. Evaluate $\int_{0.1}^{1} \left|\pi x^{-2} \sin(\pi x^{-1})\right| \mathrm{d}x$. [2]

Markscheme

a. $\sin(\pi x^{-1}) = 0 \ \frac{\pi}{x} = \pi,\ 2\pi(\ldots)$ ***(A1)***

$x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ ***A1***

[2 marks]

b. $\left[\cos(\pi x^{-1})\right]_{\frac{1}{n+1}}^{\frac{1}{n}}$ ***M1***

$= \cos(\pi n) - \cos(\pi(n+1))$ ***A1***

= 2 when *n* is even and = –2 when *n* is odd ***A1***

[3 marks]

c. $\int_{0.1}^{1} \left|\pi x^{-2} \sin(\pi x^{-1})\right| \mathrm{d}x = 2 + 2 + \ldots + 2 = 18$ ***(M1)A1***

[2 marks]

Examiners report

a. There were a pleasing number of candidates who answered part (a) correctly. Fewer were successful with part (b). It was expected by this stage of the paper that candidates would be able to just write down the value of the integral rather than use substitution to evaluate it.

b. There were a pleasing number of candidates who answered part (a) correctly. Fewer were successful with part (b). It was expected by this stage of the paper that candidates would be able to just write down the value of the integral rather than use substitution to evaluate it.

c. There were disappointingly few correct answers to part (c) with candidates not realising that it was necessary to combine the previous two parts in order to write down the answer.

If *x* satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$, show that $11 \tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^{+}$.

Markscheme

$\sin\left(x + \frac{\pi}{3}\right) = \sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right)$ ***(M1)***

$\sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 2 \times \frac{\sqrt{3}}{2}\sin x$ ***A1***

dividing by $\cos x$ and rearranging ***M1***

$\tan x = \frac{\sqrt{3}}{2\sqrt{3}-1}$ ***A1***

rationalizing the denominator ***M1***

$11\tan x = 6 + \sqrt{3}$ ***A1***

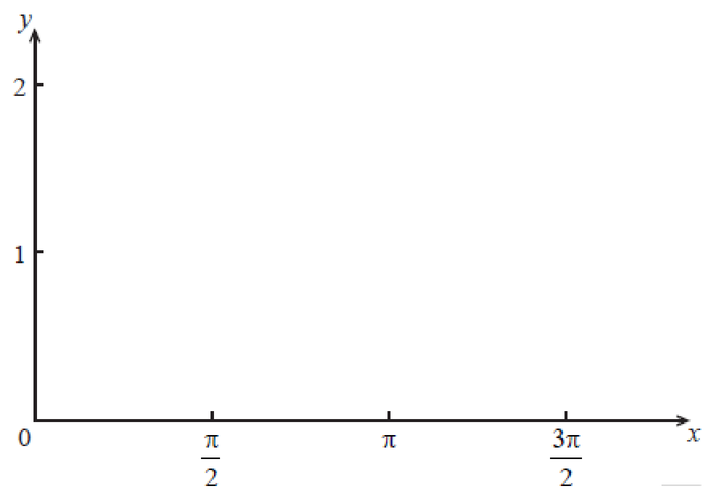
[6 marks]

Examiners report

Most candidates were able to make a meaningful start to this question, but a significant number were unable to find an appropriate expression for $\tan x$ or to rationalise the denominator.

Given that $f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

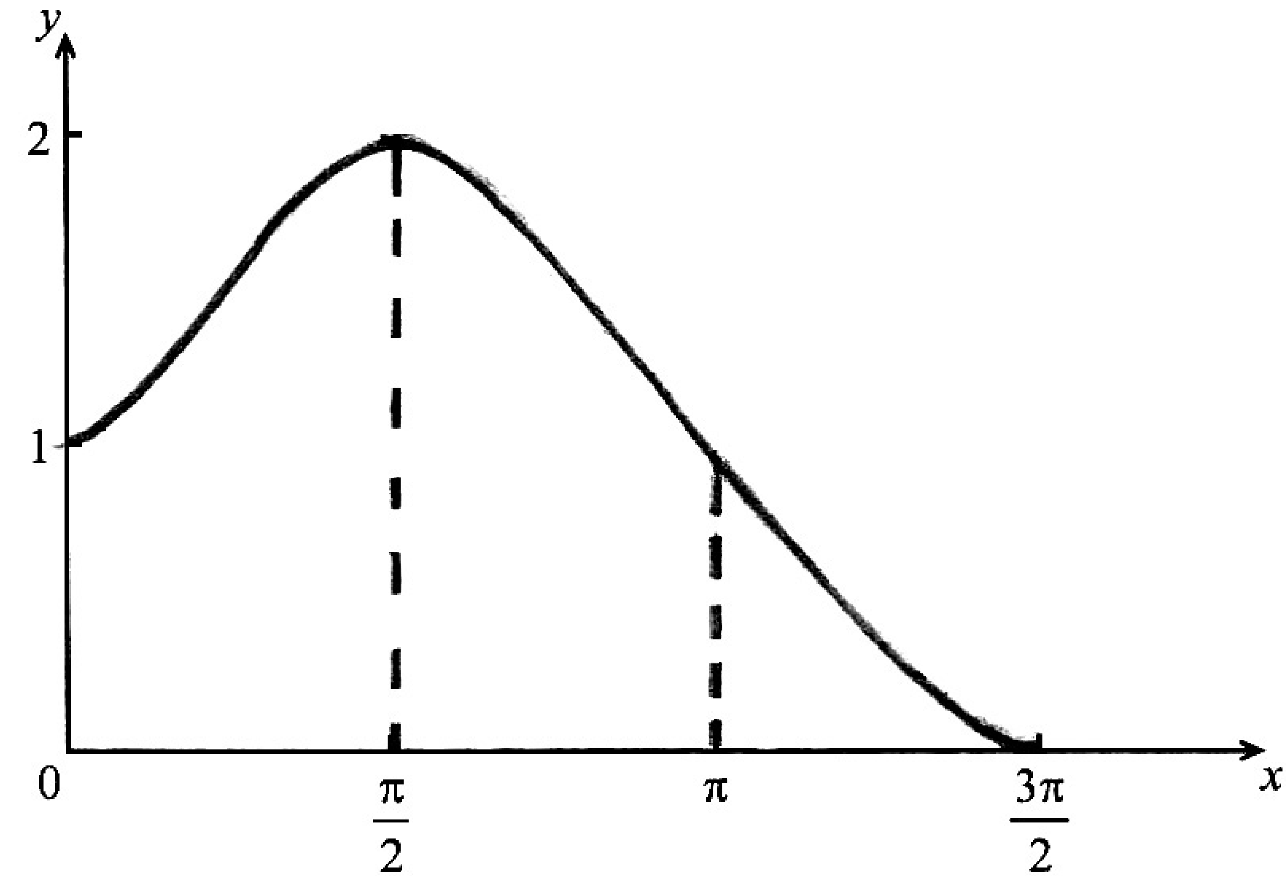
a. sketch the graph of f; [1]



b. show that $(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$; [1]

c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. [4]

Markscheme

a. 

A1

[1 mark]

b. $(1+\sin x)^2 = 1 + 2\sin x + \sin^2 x$

$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$ ***A1***

$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$ ***AG***

[1 mark]

c. $V = \pi \int_0^{\frac{3\pi}{2}} (1+\sin x)^2 \mathrm{d}x$ ***(M1)***

$= \pi \int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right) \mathrm{d}x$

$= \pi \left[\frac{3}{2}x - 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{3\pi}{2}}$ ***A1***

$= \frac{9\pi^2}{4} + 2\pi$ ***A1A1***

[4 marks]

Examiners report

a. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

b. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

c. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

The function *f* is defined on the domain $\left[0,\ \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

a. State the two zeros of *f* . [1]

b. Sketch the graph of *f* . [1]

c. The region bounded by the graph, the *x*-axis and the *y*-axis is denoted by *A* and the region bounded by the graph and the *x*-axis is denoted by *B* . Show that the ratio of the area of *A* to the area of *B* is [7]

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

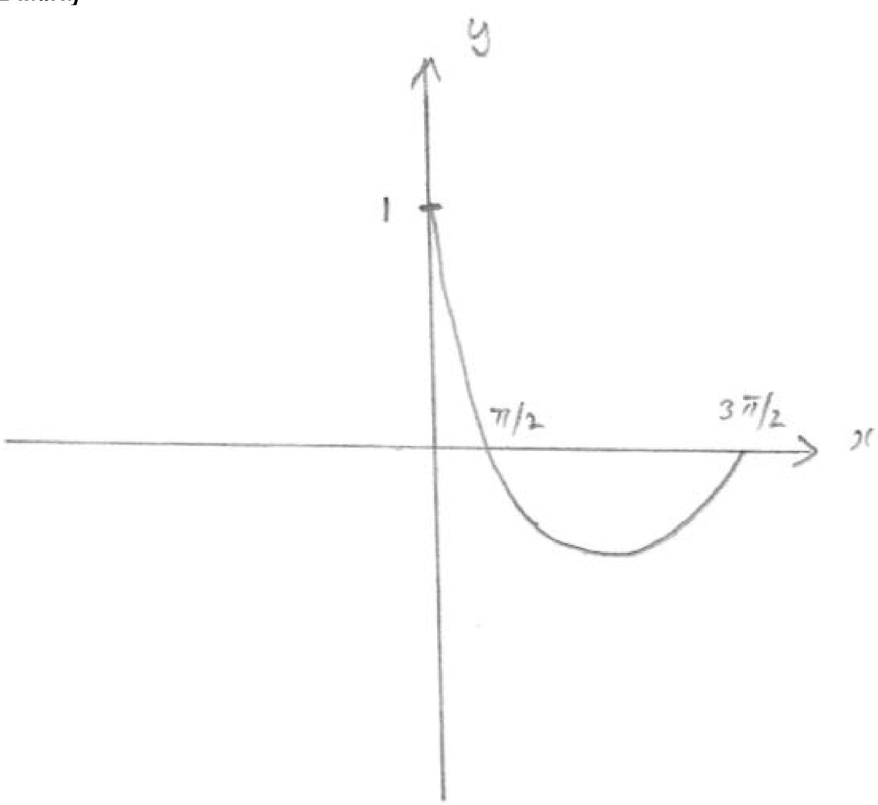
Markscheme

a. $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2},\ \frac{3\pi}{2}$ ***A1***

[1 mark]

b.



A1

Note: Accept any form of concavity for $x \in \left[0, \frac{\pi}{2}\right]$.

Note: Do not penalize unmarked zeros if given in part (a).

Note: Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

c. attempt at integration by parts ***M1***

EITHER

$I = \int \mathrm{e}^{-x}\cos x\mathrm{d}x = -\mathrm{e}^{-x}\cos x\mathrm{d}x - \int \mathrm{e}^{-x}\sin x\mathrm{d}x$ ***A1***

$\Rightarrow I = -\mathrm{e}^{-x}\cos x\mathrm{d}x - \left[-\mathrm{e}^{-x}\sin x + \int \mathrm{e}^{-x}\cos x\mathrm{d}x\right]$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

OR

$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = \mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***A1***

$I = \mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - \int \mathrm{e}^{-x} \cos x \mathrm{d}x$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

THEN

$\int_0^{\frac{\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ ***A1***

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2} - \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}$ ***A1***

ratio of *A*:*B* is $\frac{\frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}+\frac{1}{2}}{\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2}+\frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}}$

$= \frac{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{\pi}{2}}+1\right)}{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{3\pi}{2}}+\mathrm{e}^{-\frac{\pi}{2}}\right)}$ ***M1***

$= \frac{\mathrm{e}^{\pi}\left(\mathrm{e}^{\frac{\pi}{2}}+1\right)}{\mathrm{e}^{\pi}+1}$ ***AG***

[7 marks]

Examiners report

a. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

b. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

c. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

The function *f* is defined by $f(x) = \frac{1}{4x^2-4x+5}$.

a. Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. [2]

b. The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3]

c. Sketch the graph of $y = f(x)$. [2]

d. Find the range of *f*. [2]

e. By using a suitable substitution show that $\int f(x)\mathrm{d}x = \frac{1}{4}\int \frac{1}{u^2+1}\mathrm{d}u$. [3]

f. Prove that $\int_1^{3.5} \frac{1}{4x^2-4x+5}\mathrm{d}x = \frac{\pi}{16}$. [7]

Markscheme

a. $4(x-0.5)^2+4$ ***A1A1***

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

[2 marks]

b. translation $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ (allow "0.5 to the right") ***A1***

stretch parallel to *y*-axis, scale factor 4 (allow vertical stretch or similar) ***A1***

translation $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (allow "4 up") ***A1***

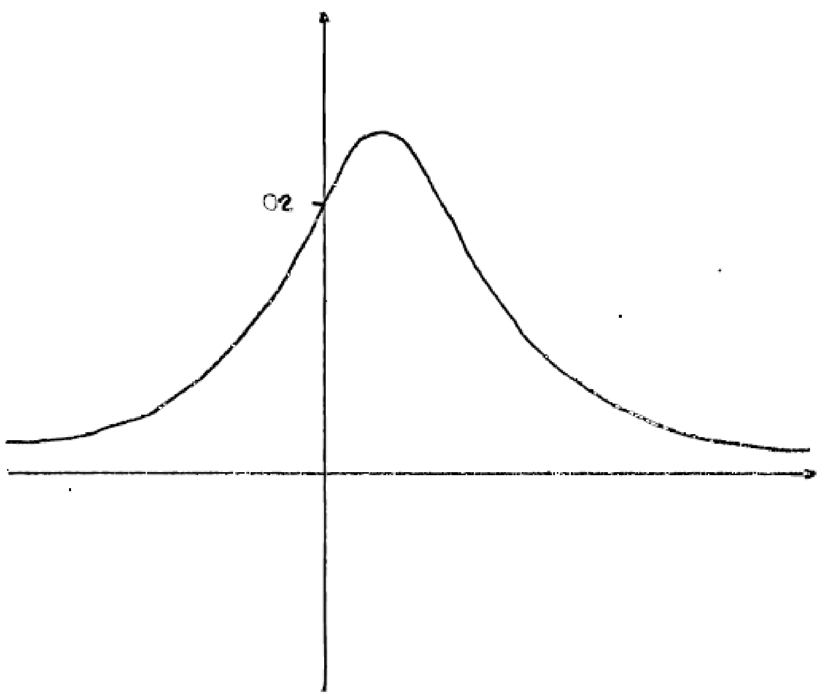
Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation of $\begin{pmatrix} 0.5 \\ 4 \end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

[3 marks]

c.



general shape (including asymptote and single maximum in first quadrant), ***A1***

intercept $\left(0, \frac{1}{5}\right)$ or maximum $\left(\frac{1}{2}, \frac{1}{4}\right)$ shown ***A1***

[2 marks]

d. $0 < f(x) \leqslant \frac{1}{4}$ ***A1A1***

Note: ***A1*** for $\leqslant \frac{1}{4}$, ***A1*** for $0 <$.

[2 marks]

e. let $u = x - \frac{1}{2}$ ***A1***

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad (\text{or } \mathrm{d}u = \mathrm{d}x)$ ***A1***

$\int \frac{1}{4x^2-4x+5}\mathrm{d}x = \int \frac{1}{4\left(x-\frac{1}{2}\right)^2+4}\mathrm{d}x$ ***A1***

$\int \frac{1}{4u^2+4}\mathrm{d}u = \frac{1}{4}\int \frac{1}{u^2+1}\mathrm{d}u$ ***AG***

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

f. $\int_1^{3.5} \frac{1}{4x^2-4x+5}\mathrm{d}x = \frac{1}{4}\int_{0.5}^{3} \frac{1}{u^2+1}\mathrm{d}u$ ***A1***

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to *x* values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}[\arctan(u)]_{0.5}^{3}$ ***(M1)***

$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***A1***

let the integral = I

$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***M1***

$\frac{3-0.5}{1+3\times0.5} = \frac{2.5}{2.5} = 1$ ***(M1)A1***

$4I = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{16}$ ***A1AG***

[7 marks]

Examiners report

a. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

b. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(b) Exam technique would have helped those candidates who could not get part (a) correct as any solution of the form given in the question could have led to full marks in part (b). Several candidates obtained expressions which were not of this form in (a) and so were unable to receive any marks in (b) Many missed the fact that if a vertical translation is performed before the vertical stretch it has a different magnitude to if it is done afterwards. Though on this occasion the markscheme was fairly flexible in the words it allowed to be used by candidates to describe the transformations it would be less risky to use the correct expressions.

c. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(c) Generally the sketches were poor. The general rule for all sketch questions should be that any asymptotes or intercepts should be clearly labelled. Sketches do not need to be done on graph paper, but a ruler should be used, particularly when asymptotes are involved.

d. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

e. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

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(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

a. (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$. [9]

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

b. Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. [8]

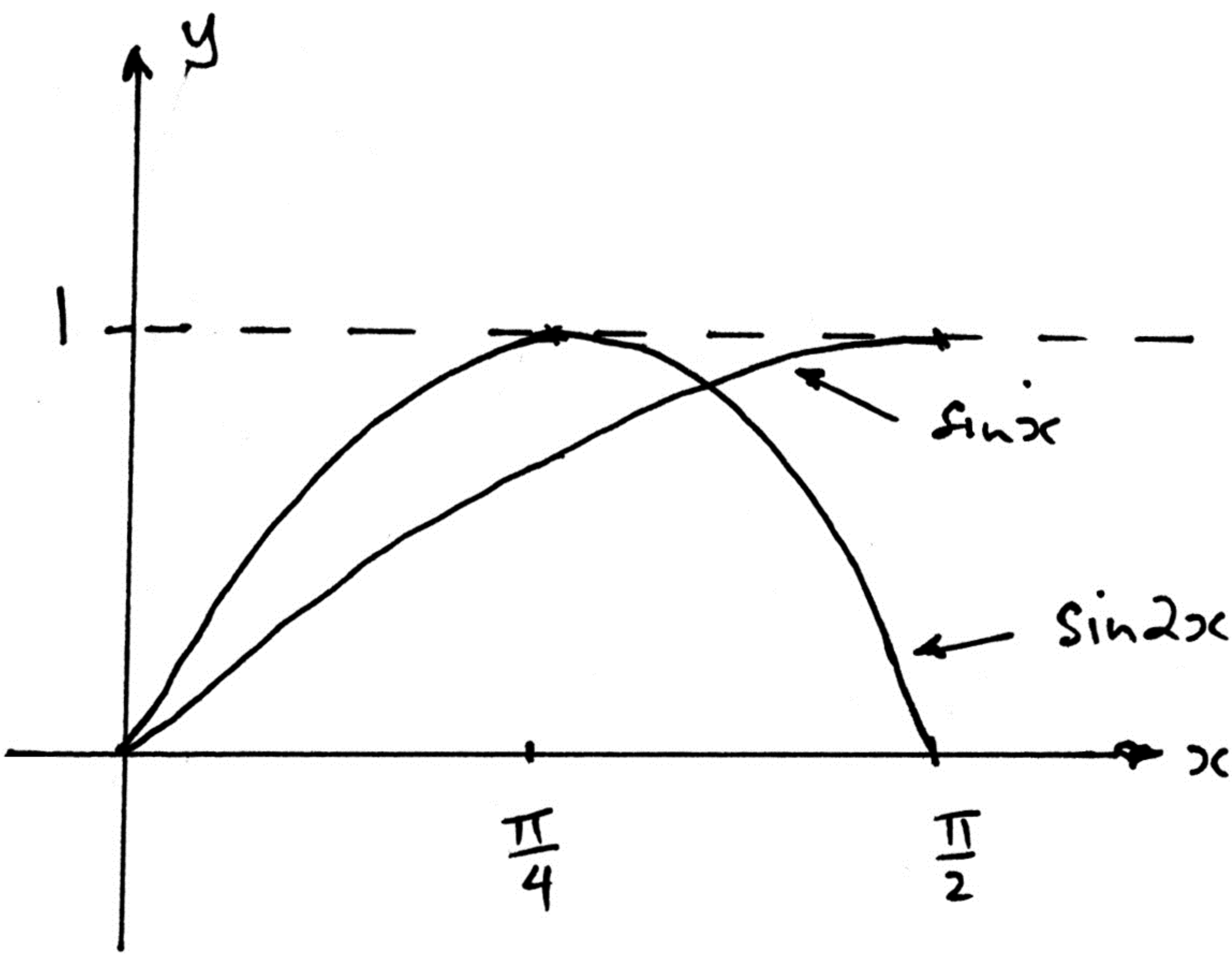
c. The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$. [8]

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

Markscheme

a. (i)



A2

Note: Award ***A1*** for correct $\sin x$, ***A1*** for correct $\sin 2x$.

Note: Award ***A1A0*** for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x$, $0 \leqslant x \leqslant \frac{\pi}{2}$

$2\sin x\cos x - \sin x = 0$ ***M1***

$\sin x(2\cos x - 1) = 0$

$x = 0, \frac{\pi}{3}$ ***A1A1*** ***N1N1***

(iii) area $= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\mathrm{d}x$ ***M1***

Note: Award ***M1*** for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ ***A1***

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ ***(M1)***

$= \frac{3}{4} - \frac{1}{2}$

$= \frac{1}{4}$ ***A1***

[9 marks]

b. $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\mathrm{d}\theta$ ***M1A1A1***

Note: Award ***M1*** for substitution and reasonable attempt at finding expression for dx in terms of dθ , first ***A1*** for correct limits, second ***A1*** for correct substitution for dx .

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\mathrm{d}\theta$ ***A1***

$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\mathrm{d}\theta$ ***M1***

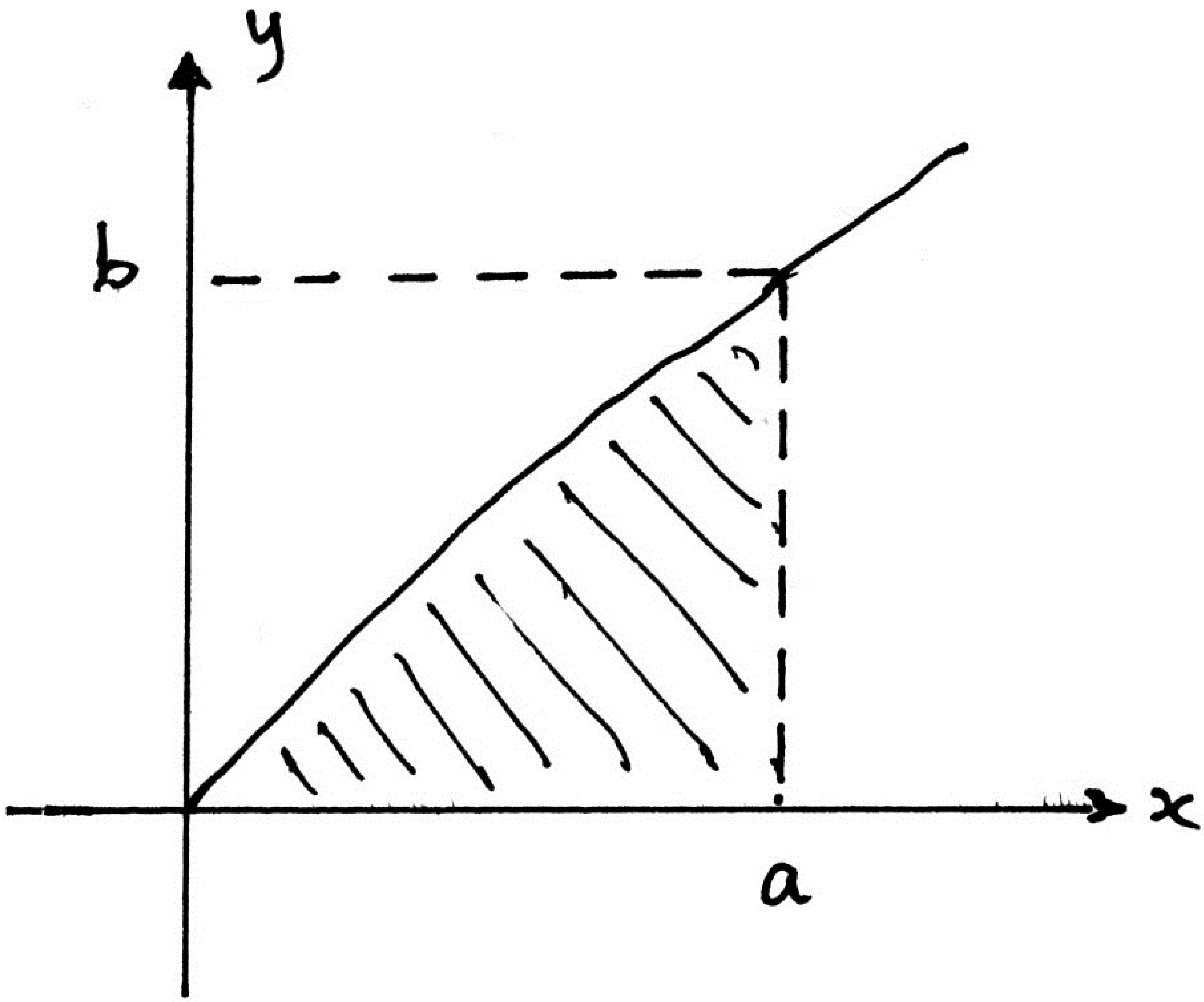
$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ ***A1***

$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ ***(M1)***

$= \frac{2\pi}{3} - \sqrt{3}$ ***A1***

[8 marks]

c. (i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$ ***R1***

$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$ ***AG***

(ii) $f(x) = \arcsin\frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ ***A1***

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\mathrm{d}x$ ***M1A1A1***

Note: Award ***A1*** for the limit $\frac{\pi}{6}$ seen anywhere, ***A1*** for all else correct.

$= \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$ ***A1***

$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ ***A1***

Note: Award no marks for methods using integration by parts.

[8 marks]

Examiners report

a. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution.
The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required.
Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

b. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

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(a) Show that $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(b) Hence, or otherwise, find the value of $\arctan(2) + \arctan(3)$.

Markscheme

(a) **METHOD 1**

let $x = \arctan\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ and $y = \arctan\frac{1}{3} \Rightarrow \tan y = \frac{1}{3}$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1$ ***M1***

so, $x + y = \arctan 1 = \frac{\pi}{4}$ ***A1AG***

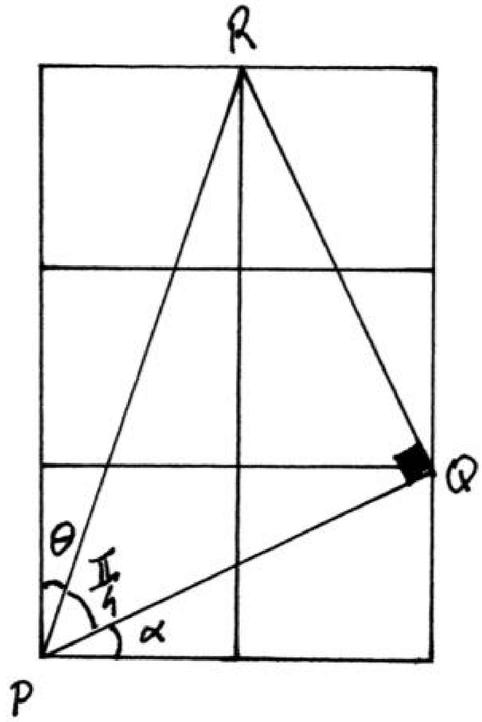
METHOD 2

for $x,\ y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ ***M1*** so, $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)=\frac{\pi}{4}$ ***A1AG***

METHOD 3

an appropriate sketch ***M1***

e.g.



correct reasoning leading to $\frac{\pi}{4}$ ***R1AG***

(b) **METHOD 1**

$\arctan(2)+\arctan(3)=\frac{\pi}{2}-\arctan\left(\frac{1}{2}\right)+\frac{\pi}{2}-\arctan\left(\frac{1}{3}\right)$ ***(M1)***

$=\pi-\left(\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)\right)$ ***(A1)***

Note: Only one of the previous two marks may be implied.

$=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ ***A1 N1***

METHOD 2

let $x=\arctan 2\Rightarrow\tan x=2$ and $y=\arctan 3\Rightarrow\tan y=3$

$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{2+3}{1-2\times 3}=-1$ ***(M1)***

as $\frac{\pi}{4}<x<\frac{\pi}{2}$ $\left(\text{accept } 0<x<\frac{\pi}{2}\right)$

and $\frac{\pi}{4}<y<\frac{\pi}{2}$ $\left(\text{accept } 0<y<\frac{\pi}{2}\right)$

$\frac{\pi}{2}<x+y<\pi$ $(\text{accept } 0<x+y<\pi)$ ***(R1)***

Note: Only one of the previous two marks may be implied.

so, $x+y=\frac{3\pi}{4}$ ***A1 N1***

METHOD 3

for $x,\ y>0$, $\arctan x+\arctan y=\arctan\left(\frac{x+y}{1-xy}\right)+\pi$ if $xy>1$ ***(M1)***

so, $\arctan 2+\arctan 3=\arctan\left(\frac{2+3}{1-2\times 3}\right)+\pi$ ***(A1)***

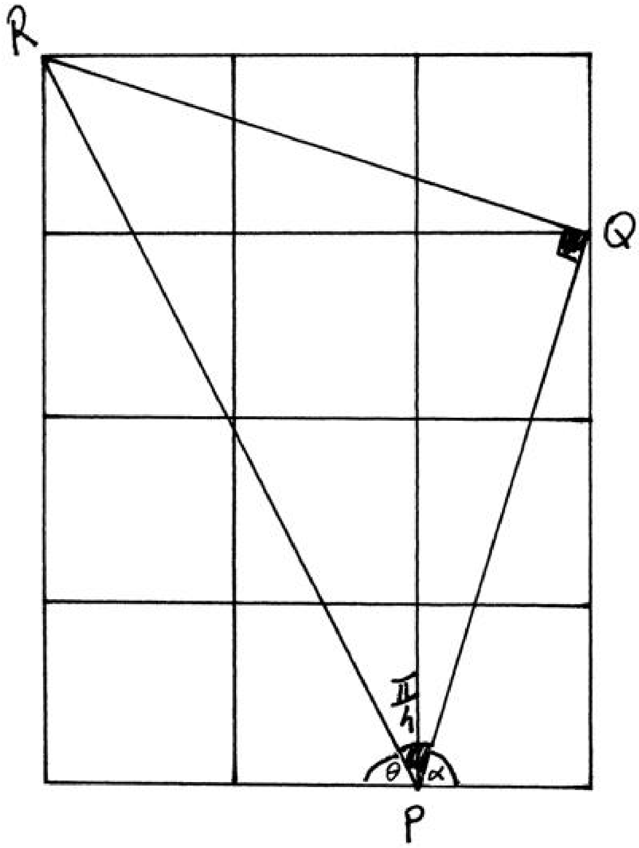
Note: Only one of the previous two marks may be implied.

$=\frac{3\pi}{4}$ ***A1 N1***

METHOD 4

an appropriate sketch ***M1***

e.g.



correct reasoning leading to $\frac{3\pi}{4}$ ***R1A1***

[5 marks]

Examiners report

Most candidates had difficulties with this question due to a number of misconceptions, including $\arctan x = \tan^{-1} x = \frac{\cos x}{\sin x}$ and $\arctan x = \frac{\arcsin x}{\arccos x}$, showing that, although candidates were familiar with the notation, they did not understand its meaning. Part (a) was done well among candidates who recognized arctan as the inverse of the tangent function but just a few were able to identify the relationship between parts (a) and (b). Very few candidates attempted a geometrical approach to this question.

(a) Sketch the curve $f(x) = \sin 2x$, $0 \leqslant x \leqslant \pi$.

(b) Hence sketch on a separate diagram the graph of $g(x) = \csc 2x$, $0 \leqslant x \leqslant \pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes.

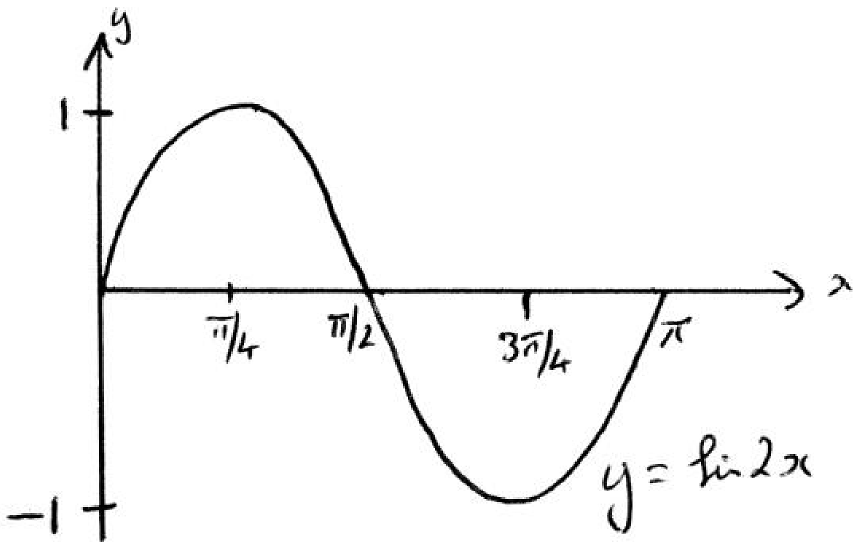
(c) Show that $\tan x + \cot x \equiv 2 \csc 2x$.

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of $y = \tan 2x + \cot 2x$, $0 \leqslant x \leqslant \frac{\pi}{2}$.

(e) Find the solution of the equation $\csc 2x = 1.5 \tan x - 0.5$, $0 \leqslant x \leqslant \frac{\pi}{2}$.

Markscheme

(a)



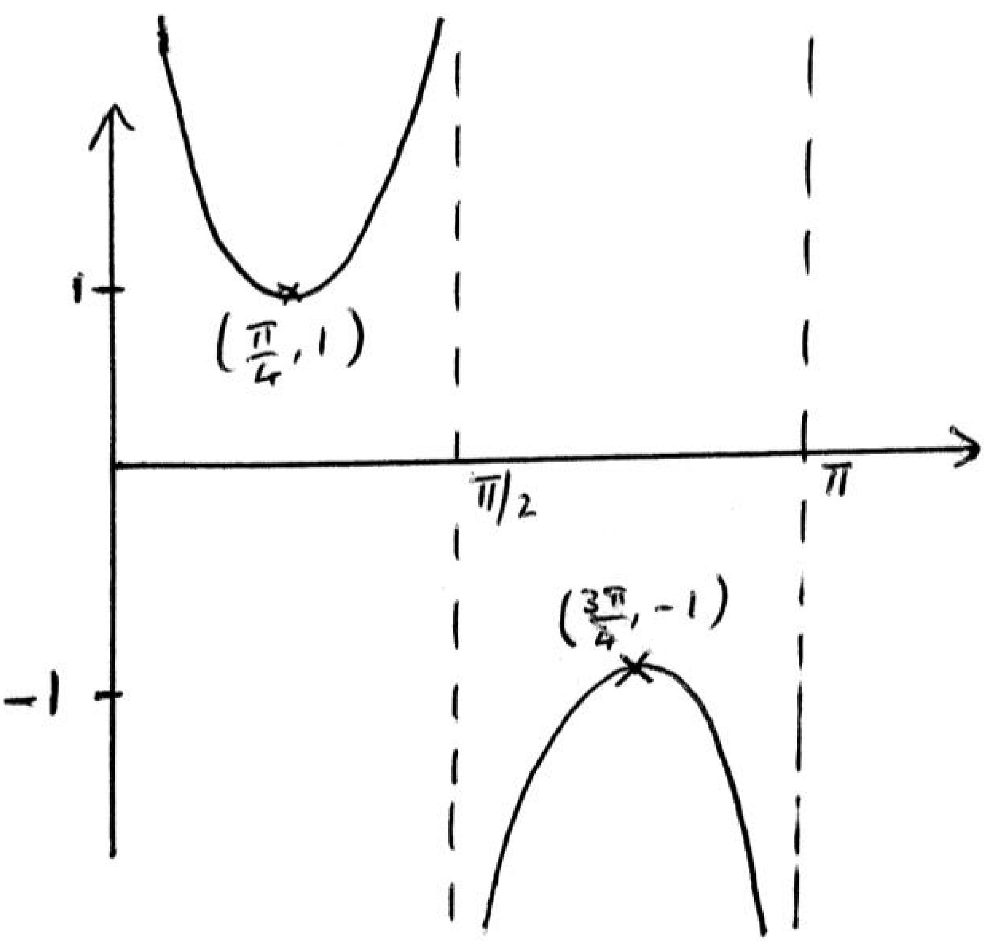
A2

Note: Award ***A1*** for shape.

A1 for scales given on each axis.

[2 marks]

(b)



A5

Asymptotes $x = 0,\ x = \frac{\pi}{2},\ x = \pi$

Max $\left(\frac{3\pi}{4}, -1\right)$, Min $\left(\frac{\pi}{4}, 1\right)$

Note: Award ***A1*** for shape

A2 for asymptotes, ***A1*** for one error, ***A0*** otherwise.

A1 for max.

A1 for min.

[5 marks]

(c) $\tan x + \cot x \equiv \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ ***M1***

$\equiv \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ ***A1***

$\equiv \frac{1}{\frac{1}{2}\sin 2x}$ ***A1***

$\equiv 2\csc 2x$ ***AG***

[3 marks]

(d) $\tan 2x + \cot 2x \equiv 2\csc 4x$ ***(M1)***

Max is at $\left(\frac{3\pi}{8}, -2\right)$ ***A1A1***

Min is at $\left(\frac{\pi}{8}, 2\right)$ ***A1A1***

[5 marks]

(e) $\csc 2x = 1.5\tan x - 0.5$

$\frac{1}{2}\tan x + \frac{1}{2}\cot x = \frac{3}{2}\tan x - \frac{1}{2}$ ***M1***

$\tan x + \cot x = 3\tan x - 1$

$2\tan x - \frac{1}{\tan x} - 1 = 0$ ***M1***

$2\tan^2 x - \tan x - 1 = 0$ ***A1***

$(2\tan x + 1)(\tan x - 1) = 0$ ***M1***

$\tan x = -\frac{1}{2}$ or 1 ***A1***

$\mathrm{x} = \frac{\pi}{4}$ ***A1***

Note: Award ***A0*** for answer in degrees or if more than one value given for *x*.

[6 marks]

Total [21 marks]

Examiners report

Although the better candidates scored well on this question, it was disappointing to see that a number of candidates did not appear to be well prepared and made little progress. It was disappointing that a small minority of candidates were unable to sketch $y = \sin 2x$. Most candidates who completed part (a) attempted part (b), although not always successfully. In many cases the coordinates of the local maximum and minimum points and the equations of the asymptotes were not clearly stated. Part (c) was attempted by the vast majority of candidates. The responses to part (d) were disappointing with a significant number of candidates ignoring the hence and attempting differentiation which more often than not resulted in either arithmetic or algebraic errors. A reasonable number of candidates gained the correct answer to part (e), but a number tried to solve the equation is terms of sin *x* and cos *x* and made little progress.